Lagrangian fibrations

Research project in collaboration

by Dimitri MARKUSHEVICH and Ugo BRUZZO

The interest in Lagrangian fibrations (LF) has several sources. **First**, the irreducible symplectic varieties (ISV) being a higher dimensional generalization of K3 surfaces, those with a LF are a generalization of K3 surfaces with an elliptic pencil. One will wonder whether the role of the LFs in the theory of ISVs is similar to that of elliptic pencils in the theory of K3 surfaces. **Second**, the LFs are, in another terminology, algebraically completely integrable systems (ACIS). So, the discovery of a new LF is at the same time a discovery of a new integrable system, pending applications in physics. **Third**, the known deformation types of ISVs being rather scarce, it is tempting to construct ISVs as fibrations in abelian varieties endowed with a symplectic 2-form making them LFs.

As concerns the **first item**, it is natural to expect, by analogy with K3 surfaces, that a rational Lagrangian fibration exists if and only if V has a divisor D with Beauville–Bogomolov square 0. The necessity to go over to *rational* LFs is due to the fact that birational ISV's have the same Beauville–Bogomolov form, and a birational transformation (flop) can destroy the regularity of a LF making it a rational map. In [M2], the question on the criteria for the existence of rational LFs on the punctual Hilbert scheme $S^{[d]}$ of a K3 surface S is addressed. It is proved that $S^{[d]}$ admits a rational LF if the Beauville–Bogomolov form on the cohomology $H^2(S^{[d]}) = H^2(S) \overset{\perp}{\oplus} \langle 2d-2\rangle$ has an isotropic vector $f + ke$, where e is a basis of $\langle 2d-2\rangle = \mathbb{Z}e$, $\mathrm{Pic}(S) = \mathbb{Z}f$ and $k \in \mathbb{Z}$. The construction of f uses a twisted Fourier–Mukai transform which induces a birational isomorphism μ between $S^{[d]}$ and a certain moduli space of twisted sheaves on another K3 surface M, obtained from S as its (twisted) Fourier–Mukai partner. The same result was obtained, simultaneously and independently, by Sawon. Later Yoshioka strengthened this result in proving that the above birational isomorphism μ is biregular, so that the LF constructed on $S^{[d]}$ is not just rational, but regular. However, if $\mathrm{Pic}(S) \not\simeq \mathbb{Z}f$, an example given in [M2] shows that f may have a nonempty indeterminacy locus.

We plan to extend this result to the general case when the Beauville–Bogomolov form represents zero on $H^2(S^{[d]})$, that is, the case when there exists an isotropic vector $v \in H^2(S^{[d]})$ of the form $kf + le$ with $k, l \in \mathbb{Z}$ mutually prime, $f \in H^2(S)$ a primitive class. The next question to investigate is whether the orbits of such classes $kf + le$ under the monodromy (resp. the full orthogonal) group action exhaust all the integer isotropic vectors in $H^2(S^{[d]})$. A positive answer would mean that for each isotropic vector $v \in H^2(S^{[d]})$, the hyperplane section $v^\perp$ of the period domain has a dense open subset of points representing periods of regular LFs given by the linear system of the divisor class v. Stated in this form, the result will imply a complete analogy with the description of periods of K3 surfaces having an elliptic pencil with class v.

Quite naturally, we will also think about possible new explicit constructions of Lagrangian fibrations that will extend this kind of results to other moduli spaces of ISVs, and generalized ISVs with mild singularities (see the third item below).

As concerns the **second item**, there are rather few known Lagrangian fibrations on ISVs. Generally speaking, they all are obtained by finding a birational isomorphism between the given ISV M and some torsor T under the compactified relative Jacobian J of a complete linear system $|C|$ of genus-g curves on a K3 surface, or else between M and the fiber of the Albanese map $T \to \mathrm{Alb}(T) = A \times \hat{A}$, where T is some torsor under the compactified relative Jacobian J of a linear system of curves on an abelian surface A (see [M2], [Gu], [De] and references therein). The compactified relative Jacobian is understood here as the moduli space of Simpson-semistable torsion sheaves with Euler number $1-g$, supported on curves from $|C|$ and having rank 1 when considered as sheaves on their support. The LF map is the support map, sending each sheaf to its support; its base is always the projective space $|C| = \mathbb{P}^g$ for a K3 ($\mathbb{P}^{g-2}$ for an abelian) surface, and the generic fiber is the Jacobian of a genus-g curve for a K3 surface (or its abelian subvariety of codimension 2 with polarization $(1, \ldots, 1, g-1)$ for an abelian surface).

On the other hand, there are examples of LFs over a noncompact base. These are the Donagi–Markman integrable system [DoM] of the intermediate Jacobians of the hyperplane sections of a smooth cubic 4-fold in $\mathbb{P}^5$, and the integrable system of intermediate Jacobians of the complete family of gauged Calabi–Yau manifolds [DoM] or of the complete family of Fano varieties with fixed hyperplane section [B], [IM1], [IM2], [M3]. The case considered in [M3] is that of the relative intermediate Jacobian of the moduli space of the K3–Fano flags $S \subset X$, where S is a fixed K3 surface and X a Fano 3-fold containing S as an anticanonical divisor.

The fibers of these LFs are in general not Jacobians of curves, and it is a very interesting question whether their total space can be compactified into a smooth symplectic variety on which the LF remains regular. If yes, this will give first examples of LFs on ISVs whose fibers are principally polarized, but are not Jacobians of curves. The first case to look at is that of the K3–Fano flag in which S is a degree-6 K3 surface and X is a cubic 3-fold, first considered in [B]. As show [B] and a recent preprint of Hwang–Nagai, this LF is closely related to the sporadic 10-dimensional ISV of O'Grady. This problem might be treated for more types of K3–Fano flags (say, for some flags with Fano of index 1) in the case if we suggest this theme to a PhD student.

Another question that should be treated is that of a description of the bases of the LFs of [M3]. A priori, they are Deligne–Mumford stacks, but are expected to be open sets of projective spaces.

Another way to associate an integrable system to K3-Fano flags is to take a pair of K3-Fano flags (S, V_1), (S, V_2) with the same S, and consider the normal crossing variety $X = V_1 \cup V_2$, where V_1, V_2 intersect transversely along S. Then X can be smoothed to a Calabi–Yau threefold. The Donagi–Markman integrable system over the moduli space of gauged Calabi–Yau threefolds obtained by smoothing X defines the "boundary" integrable system whose Liouville tori would be the products $J(V_1) \times J(V_2) \times J(C)$, where C is a curve running through the linear system of the line bundle $\mathcal{N}_{S/V_1} \otimes \mathcal{N}_{S/V_2}$. A similar construction was described in [Do], Section 6, where the normal crossing Calabi–Yau was the union of two threefolds, each an elliptic fibration over a del Pezzo surface rather than a Fano threefold, and the fibers of the boundary integrable system were just the products $J(V_1) \times J(V_2)$. We plan to write down the technical details justifying this construction and to determine the heterotic counterpart of the boundary integrable system in the framework of the heterotic/F-theory duality described in loc. cit.

As concerns the **third item**, we plan to continue the work, started in [MT] and aimed at new constructions of generalized ISVs with certain mild singularities (with a hope that one may occasionally get nonsingular ones, too). This theme is extremely convenient for a work with a PhD student and will be treated to a larger extent if there is one. Narrowly understood, this is the study of the relative compactified Prym varieties of τ-invariant parts of linear systems of curves on K3 surfaces with an anti-symplectic involution. In a larger sense, these are components of the fixed loci of finite groups of *symplectic* automorphisms of holomorphically symplectic varieties (the compactified Pryms are particular cases of these). One may also consider fibrations in generalized Prym–Tyurin varieties as those studied in the recent work of Izadi–Lange and try to find out, which of them admit a symplectic structure making them LFs.

The case treated in loc. cit. is that of a quartic K3 surface S with an anti-symplectic involution τ. The relative compactified Prym P is singular for generic (S, τ), having 28 singular points of type $\mathbb{C}^4/-1$, and we introduce a notion of an irreducible symplectic V-manifold (ISVM) so that P is an ISVM. There is some evidence that any relative compactified Prym of polarization $(1, 2)$ which admits a structure of a LF is obtained in this way. This would mean, that there are no nonsingular ISVs with LFs of polarization $(1,2)$; remind that the known examples of LFs on nonsingular ISVs of dimension 4 have polarizations either $(1,1)$or $(1,3)$.

The following questions will be investigated :

– Describe the compactified relative Prym variaties associated to the linear systems on double covers S of a Del Pezzo surface D with branch divisor in $|-2K_D|$. One should first look at the linear systems of inverse images in S of curves on D of genera 1 and 2.
– Study the relative compactified Prymians P of genus-g linear systems on a generic Enriques surface E for small values of $g = 2, 3, \dots$. For $g = 2$, P should be a K3 surface, and it is interesting to find its relation to the K3 double cover S of E. Probably, P is a Fourier-Mukai partner of S. For $g = 2$, P has a LF in principally polarized abelian surfaces which is not constructed as the relative Jacobian of a family of genus-2 curves. However, it should be possible to represent it as the relative Jacobian of a linear system of genus-2 curves on some K3 surface S', and the question is to find S' ; the latter might be a Fourier–Mukai partner of S.
– Prove the above conjecture on the nonexistence of smooth LFs with polarization $(1,2)$.
– Compute the Hodge numbers and the Beauville–Bogomolov form for the three ISVMs P, P^2 (a torsor under P) and M (obtained by a flop from P) constructed in [M1]. It is interesting to compare their cohomology lattices H^2 to those studied by Gritsenko–Hulek–Sankaran. Probably, these ISVMs will provide some of the lattices studied by these authors with a modular interpretation ; if not, one gets a new type of period domains to study. This question can be extended to other ISVMs that will be discovered in the work on the previous points of the program.

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[M3] Markushevich, D. : *An integrable system of K3-Fano flags*, math.AG/0703166, to appear in Mathematische Annalen.

[MT] Markushevich, D., Tikhomirov, A. S. :*New symplectic V-manifolds of dimension four via the relative compactified Prymian*, International Journal of Mathematics **18**, 1187 – 1224 (2007).

Curriculum Vitae

- Name : MARKUSHEVICH (MARKOUCHEVITCH)
- First Name : Dimitri
- Birth date: 11 July 1960
- Sex: Male

- Degrees :

1995 Habilitation : University Lyon 1
1985 PhD : Moscow State University
1982 Diploma of 5 years of university studies from Moscow State University

- Current position : Professor at the University Lille 1

- Professional Experience:

1996-pres.	Professor at the University of Lille I
1993-1996	Senior Lecturer at the University of Lyon I
1991-1993	Senior Lecturer at the Technion - Israel Institute of Technology
1985-1991	Researcher, Institute of Mechanics of the Academy of Sciences of the USSR

- Other professional activities since 2003:
  - 2006/2007 Guest Professor at Max-Planck-Institut für Mathematik in Bonn (1 year)
  - 2007 Visitorship in the Mittag-Leffler Institute, Stockholm (1 month) and SISSA in Trieste (1 month)
  - Head of the Algebraic Geometry and Arithmetic group of the Paul Painlevé Laboratory UMR CNRS 8524 (from 2006/2007).
  - Member of the Scientific Prospecting Committee of the Paul Painlevé Laboratory UMR CNRS 8524 (from 2006/2007).
  - Editor (jointly V. B. Matveev, A. Treibich) of the Proceedings of the conference *Spectral Curves and Integrable Systems*, held in Lille in June 2003, Lett. Math. Phys. **76**, no. 2-3 (2006).
  - Member of the Hiring Committee of the Math. Department of the University of Lille (2001-2004 and 2007-present)
  - Organization of the weekly seminar on Algebraic Geometry in the University of Lille (until the end of 2005/2006).
  - Co-organisation with F. Laytimi and J. Nagel of the conference "Journées de Géométrie Algébrique" in Lille, May/2005.
  - RiP stay - "Research in pairs", 3 weeks in 2004 in Mathematisches Forschungsintstitut Oberwolfach for collaboration with A. Iliev (Sofia, Bulgaria)
  - Co-organisation with V. Gritsenko of the conference "Geometry of moduli spaces", Lille, June/2003.
  - Co-organisation with A. Treibich of the conference "Integrable systems and spectral curves", Lille, June/2003

Publications : 44 in refereed titles, more than 200 citations.

The list of papers in the last 5 years:

[1] *A parametrization of the theta divisor of the quartic double solid*, jointly A. S. Tikhomirov, Int. Math. Res. Notices **2003**, 2747–2778.
[2] *Symplectic structure on a moduli space of sheaves on a cubic fourfold*, jointly A. S. Tikhomirov, Izv. Ross. Akad. Nauk Ser. Mat. **67**, 131–158 (2003).
[3] *Elliptic curves and rank-*2 *vector bundles on the prime Fano threefold of genus* 7, jointly A. Iliev, Advances in Geometry **4**, 287–318 (2004).
[4] *Rational Lagrangian fibrations on punctual Hilbert schemes of K3 surfaces*, Manuscripta Math. **120**, no. 2, 131–150 (2006).
[5] *Parametrization of* $\operatorname{Sing}\Theta$ *for a Fano* 3*-fold of genus* 7 *by moduli of vector bundles*, jointly A. Iliev, Asian J. of Math. **11**, 427–458 (2007).
[6] *New symplectic* V*-manifolds of dimension four via the relative compactified Prymian*, jointly A. S. Tikhomirov, Intern. J. of Math. **18**, 1187 – 1224 (2007).
[7] *An integrable system of K3-Fano flags*, math.AG/0703166, to appear in Math. Annalen.
[8] *Symplectic structures on moduli spaces of sheaves via the Atiyah class*, jointly A. Kuznetsov, preprint, 2007, MPIM-2007/32 = math.AG/0703264.

Experience of project coordination:

Coordinator of the French team in the INTAS cooperative networks "Geometry and Topology of Algebraic Varieties":

- INTAS-OPEN-97-2072 (1997-2000)
- INTAS-OPEN-2000-269 (2001-2003)

The nodes of these networks:

Bar-Ilan University, Israel (Mina Teicher, global coordinator) University of Lille (Markushevich, coordinator of the French team)
University of Nice (Beauville)
University of Nancy (Campana)
Steklov Institute in Moscow (Vik. Kulikov, coordinator of the Russian team)
Moscow State University (V.A.Iskovskikh)
Yaroslavl State Pedagogical University (A.S.Tikhomirov)
Moscow Independent University (S.A.Kuleshov)

A LETTER OF REFERENCE IN SUPPORT OF DIMITRI MARKUSHEVICH'S APPLICATION FOR A VISIT AT SISSA IN TRIESTE

4 March, 2008

Dimitri Markushevich has asked me to be be an endorser of his application for a stay at SISSA, and I am glad to comment on his research. He is an author of more than 50 papers, so I will not mention all the various results contained in them. I will write only on a part of his research close to my interests. This is the field of holomorphically symplectic manifolds. He has publications starting from 1985 in this field, so I will dwell upon the most recent of them, starting from 2006. These are important results on Lagrangian fibrations on the holomorphically symplectic manifolds and a construction of holomoprhic 2-forms over moduli spaces of sheaves via the Atiyah class.

In the paper D. Markushevich: *Rational Lagrangian fibrations on punctual Hilbert schemes of K3 surfaces*, Manuscripta Math. **120**, no. 2, 131–150 (2006), the question on the criteria for the existence of rational Lagrangian fibrations on the punctual Hilbert scheme $S^{[d]}$ of a K3 surface S is addressed. This question is of particular importance by the reason that, first, a Lagrangian fibration is a synonym for an algebraically completely integrable system, and second, a Lagrangian fibration is a higher-dimensional analog of an elliptic pencil on a K3 surface, a key object for study of many problems about K3 surfaces.

A rational Lagrangian fibration f on an irreducible symplecitc variety V is a rational map which is birationally equivalent to a regular surjective morphism with Lagrangian fibers. By the mentioned analogy with K3 surfaces, it is natural to expect that a rational Lagrangian fibration exists if and only if V has a divisor D with Bogomolov–Beauville square 0. This conjecture is proved in the case when V is $S^{[d]}$ for a generic algebraic K3 surface S. The construction of f uses a twisted Fourier–Mukai transform which induces a birational isomorphism μ between $S^{[d]}$ and a certain moduli space of twisted sheaves on another K3 surface M, obtained from S as its (twisted) Fourier–Mukai partner. The same result was obtained, similtaneously and independently, by Sawon. Yoshioka strengthened this result in proving that the above birational isomorphism μ is biregular, so that the Lagrangian fibration constructed on $S^{[d]}$ is not just rational, but regular (Yoshioka's proof is included in the paper J. Sawon: *Lagrangian fibrations on Hilbert schemes of points on K3 surfaces*, J. Algebraic Geom. **16**, 477–497 (2007)). However, if S is not generic, an example given by Markushevich shows that f may have a nonempty indeterminacy locus.

In D. Markushevich, A. S. Tikhomirov: *New symplectic V-manifolds of dimension four via the relative compactified Prymian*, International Journal of Mathematics **18**, 1187 – 1224 (2007), three new examples of 4-dimensional irreducible symplectic V-manifolds are constructed. Two of them are relative compactified Prymians of a family of genus-3 curves with involution, and the third one is obtained from a Prymian by Mukai's flop. They have the same singularities as two of Fujiki's examples, namely, 28 isolated singular points analytically equivalent to the Veronese cone of degree 8, but a different Euler number. The family of curves used in this construction forms a linear system on a K3 surface with involution. The structure morphism of both Prymians to the base of the family is a Lagrangian fibration in abelian surfaces with polarization of type (1,2). This is the first example of a Lagrangian fibration of such polarization, and no example is known on nonsingular irreducible symplectic varieties.

In Markushevich D.: *An integrable system of K3-Fano flags*, math.AG/0703166, to appear in Mathematische Annalen, given a K3 surface S, the author shows that the relative intermediate Jacobian of the universal family of Fano 3-folds V containing S as an anti-canonical divisor is a Lagrangian fibration. This can be thought of as a logarithmic version of the Donagi–Markman integrable system of the universal family of gauged Calabi–Yau threefolds, a K3-Fano flag being a kind of a logarithmic Calabi–Yau. The proof uses variations of the mixed Hodge structure of the pair (V, S). This provides a construction of around 50 new integrable systems pending further study and physical applications.

In Kuznetsov A., Markushevich D.: *Symplectic structures on moduli spaces of sheaves via the Atiyah class*, MPIM-2007/32 = math.AG/0703264, it is proved that the composition of the Yoneda coupling with the semiregularity map is a closed 2-form on moduli spaces of sheaves. This is a general construction of closed 2-forms on moduli spaces of sheaves over a smoth projective variety using the Atiyah class of the sheaves. The authors show that it provides the symplectic structure on a 10-dimensional moduli space $P(Y)$, which is conjecturally deformation equivalent to the symplectic 10-fold of O'Grady, as well as the symplectic structure of Beauville–Donagi on the 4-dimensional variety of lines $F(Y)$. Both of these examples are moduli spaces of torsion sheaves on a cubic 4-fold Y.

While the closedness of the 2-form is a general phenomenon, its nondegeneracy is proven in a way specific for the cubic 4-fold. It turns out that the sheaves from $F(Y)$ and $P(Y)$ belong to a subcategory $C(Y)$ of the derived category $D(Y)$ which is a deformation of the derived category of a K3 surface. All the moduli spaces parametrizing sheaves from $C(Y)$ are symplectic by the same reason by which the moduli spaces of simple sheaves on a K3 surface do.

The research proposal of D. Markushevich contains interesting ideas. The project on Lagrangian families of Prym varieties is conceived to be worked on in an interaction, if not in a collaboration, with Ugo Bruzzo. Of course, as Dimitri Markushevich proposes only a short visit, there may be not sufficient time at the SISSA to finish the mentioned project, but he will certainly benefit for its advancement from his visit.

To conclude, Dimitri Markushevich is a renowned specialist in the field of holomorphically symplectic manifolds. He obtained important and beautiful results, applied a very large range of techniques. I met him on many occasions, in several mathematical conferences, and it was always a pleasure to discuss mathematics with him. He is very active in his research work and has interesting research projects. I strongly support his application for a visit in the SISSA.

Christoph Sorger
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Trieste, 4 March 2008

Object: Declaration to accompany Prof. D. Markouchevitch application for an extended MISGAM grant

Dear Sirs,

In connection with Prof. D. Markouchevitch's application for a MISGAM grant to be spent at this School, and on behalf of the Mathematical Physics Sector of the School, I am pleased to confirm that the Sector agrees to host Prof. Markouchevitch during the period Jun 24 to July 31, 2008.

Sincerely,

Ugo Bruzzo