PROPOSED RESEARCH AND AIM OF THE VISIT TO THE UNIVERSITY OF VIENNA

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The purpose of the proposed visit is research in collaboration concerning the problem that we refer to as the

GENERALIZED TODA SHOCK/RAREFACTION PROBLEM.

It is a natural extension of three recent papers [1], [2], [3] (also funded by MISGAM) concerning short range perturbations of the (quasi)periodic Toda lattice.

In the three papers above we have considered the stability of the periodic Toda lattice under a short range perturbation. More precisely, let the purely periodic Toda lattice (of period N) be given by the equations

$$\begin{aligned} \dot{b}_n^p &= 2(a_n^p)^2 - 2(a_{n-1}^p)^2, \\ \dot{a}_n^p &= a_n^p(b_{n+1}^p - b_n^p), \\ a_{N+n}^p &= a_n^p, \quad b_{N+n}^p = b_n^p. \end{aligned}$$

Consider also the doubly infinite Toda lattice

$$\begin{aligned} \dot{b}_n &= 2(a_n^2 - a_{n-1}^2), \\ \dot{a}_n &= a_n(b_{n+1} - b_n), \\ n &\in \mathbb{Z}, \end{aligned} \tag{1}$$

with initial data such that the first moment of the difference to the periodic lattice is finite

$$\sum_n |n|(|a_n - a_n^p| + |b_n - b_n^p|) < \infty$$

at time $t = 0$. The question is the behavior of a_n, b_n when $t \to \infty$.

In [1] and [2] we have shown that the limiting behavior is described by a particular modulated lattice. Our method of proof is based on an extension of the (nonlinear) stationary phase method for Riemann-Hilbert problems, to the case where the underlying space is a hyperelliptic curve.

In [3], we are studying the next order terms. In particular, we extract the term of order $t^{-1/2}$ that results from the solution of a "local" Riemann-Hilbert problem across a small cross set on a hyperelliptic curve. That problem reduces to the solution of the parabolic cylinder ODE.

Our proposed project is to consider the doubly infinite Toda lattice (1) under data that are short perturbations of different finite gap solutions at $+\infty$ and $-\infty$, not necessarily of the same (positive) genus.

The starting point of our analysis is the recently constructed inverse scattering theory of [4]. It enables us to write the inverse scattering problem as a Riemann-Hilbert factorization problem on a Riemann surface of positive genus. We will then use and generalize our recent work [2] which constructs a nonlinear stationary phase theory for Riemann-Hilbert problems on hyperelliptic curves.

The problem posed is a generalization of the Toda shock problem (first studied by Holian, Flaschka, McLaughlin in the 1980s and further analyzed in [5], [6]) and the Toda rarefaction problem studied in [7]. (In these simple cases the genus is either 1 or 0, at both infinities.) We recall that in [7] we had shown that the rarefaction problem reduces to the analysis of orthogonal polynomials. It will be interesting to see what the positive genus analogue will be.

The generalized Toda shock/rarefaction problem we propose analyzing is a dispersive shock; a discrete (in space) analogue of the semiclassical/zero-dispersion problem for nonlinear integrable PDEs ([8]). So, we expect to encounter all the phenomena (caustics, Whitham equations, high frequency oscillations, universality at the caustics, etc. [8], [9]) that one encounters in such problems, and, hopefully a nontrivial generalization thereof.

At first stage, we will consider the real Toda lattice, so that the associated Lax operator is self-adjoint. At a later stage we intend to consider the Ablowitz-Ladik lattice, recently studied in [10], where the Lax operator is non-self-adjoint and the analysis is more delicate (see e.g. [11], [12]).

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