Final report for a MISGAM short visit grant

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Numerical study of the small dispersion limit of the Korteweg-de Vries equation near singular points

The solution of the Cauchy problem for the Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0, \quad u(x,0) = u_0(x), \tag{1}$$

in the small dispersion limit $\epsilon \to 0$ is known to form an oscillatory zone which is asymptotically described by the theory of Lax and Levermore [LL], Venakides [V2] and Deift, Venakides and Zhou [DVZ]. In [GK] we have presented a quantitative numerical comparison between the solution of the KdV equation and the corresponding asymptotic solution. It was shown that the asymptotic description of [LL], [DVZ] gives in general a good approximation of the KdV solution, but is less satisfactory near the point of gradient catastrophe of the hyperbolic equation, $u_t + 6uu_x = 0$ [D] and at the boundary of the oscillatory region being formed after this critical time.

In this project we have addressed the asymptotic description of the KdV equation in the small dispersion limit at the leading edge of the oscillatory zone. An enhanced asymptotic description is obtained via a multi-scales expansion of the KdV solution near the leading edge which is given in terms of a special solution of the Painlevé II equation. This solution was constructed by considering an asymptotic expansion to set up a boundary value problem on a finite interval. The resulting problem is then solved numerically with a Chebyshev collocation method and a strongly relaxed fixed point iteration. We thus obtain the solution with a precision of at least 10^{-5}. By comparing numerically the multi-scales expansion with the KdV solution we have shown that the latter in fact provides a considerably better description than the LL and DVZ theory near the leading edge: whereas the latter gives an error decreasing as $\epsilon^{1/3}$, the former yields an error of the order $\epsilon^{2/3}$ as predicted. We identify numerically the zone where the multi-scales solution provides a better description than the LL and DVZ theory and show that it scales as expected as $\epsilon^{2/3}$.

In a future collaboration with the host institution, the multi-scales expansion will be carried out to higher order to obtain a description of the order ϵ. The results obtained during this visit will be published in a forth-coming publication.

[DVZ] P. Deift, S. Venakides, and X. Zhou, *New result in small dispersion KdV by an extension of the steepest descent method for Riemann-Hilbert problems*, IMRN **6**, (1997), 285-299.

[D] B. Dubrovin, *On Hamiltonian Perturbations of Hyperbolic Systems of Conservation Laws, II: Universality of Critical Behaviour*, Comm. Math. Phys., **267** (2006), 117.

[LL] P. D. Lax and C. D. Levermore, *The small dispersion limit of the Korteweg de Vries equation, I,II,III*, Comm. Pure Appl. Math. **36** (1983), 253-290, 571-593, 809-830.

[V] S. Venakides, *The Korteweg de Vries equations with small dispersion: higher order Lax-Levermore theory*, Comm. Pure Appl. Math. **43** (1990), 335-361.