Report MISGAM Exchange Grant 1660: Extending 2D multi-component Toda hierarchies

August 20, 2008

1 Purpose of the visit

The aim of the visit was to work on the possibility of finding new extended flows within the context of the the multicomponent 2D Toda hierachy. These flows have been discussed in [4] where for a particular reduction of the 1-component 2D Toda hierarchy an infinite set of extended flows were found. The construction, motivated by the search of integrable structures in the theory of Gromov–Witten invariants, is based on the consistent definition of logarithmic functions of Lax operators –which in turn are shift operators–. The construction has been generalized to bigraded Toda hierarchies in [3], where not only new logaritmic flows but also root type flows were proposed.

2 Description of the work carried out during the visit

The work during the visit may be divided into different parts. In the one hand, we will refer to the one regarding the discussions with the group of Prof. Dubrovin on the different descriptions of the multicomponent Toda hierarchy: factorization problem, Lax equations, Zakharov–Shabat equations, reductions and dispersionless limits. The talk delivered by the author was the starting point of this interchange of ideas. A cornerstone was the re-

cently discovered role of the multicomponent KP theory as a dispersionfull Whitham hierarchy, see [6]. These discussions went on on two principal lines:

1. With Mattia Caffaso. Reductions of multicomponent Toda hierarchy of block Toeplitz bi-infinite type and possible developments in the infinite case (not bi-infinite case). Possible relation of the block Topelitz reduction with multiple orthogonal polynomials. See [2] for the discussion for the two-component case. Also the references [1] are relevant in this context.

2. With Guido Carlet. Reductions of multicomponent Toda hierarchy of block Hankel bi-infinite type and the dispersive Whitham hierarchy. Possible extended flows for these last hierarchies.

In both cases the discussions are in a preliminar stage an need of further developements.

On the other hand, and as a continuation of the work performed in previous visit to Prof. van Moerbeke in Louvain, there is the work on multicomponent integrable systems, extending the results of the papers [7] as we will describe in the following section. We also worked out the Takasaki-Takebe finding of the dispersionfull Whitham hierarchy within the multicomponent Toda scheme.

3   Description of the main results obtained

For the multicomponent Toda hierarchy we have discussed some reductions which might be relevant in the dispersionless limit. In particular

- Block Toeplitz/Hankel reductions

- Periodic reductions

- Takasaki–Ueno reductions

For the dispersive Whitham hierarchy

- Derivation of the universal genus 0 Whitham hierarchy

- Multicomponent dispersionless Toda hierarchy: Alternative dispersionless limit in which the dToda hierachy is included naturally.

- Corresponding dispersive string equations: Derivation of the dispersionless string equations of [8] from the diagonal string equations in the multicomponent Toda hierarchy. Here the block Toeplitz/Hankel reduction plays a very important.

4 Future collaboration with host institution

A number of problems subject to collaboration remain open

1. Analysis of the role of the multicomponent KP solutions of [1]:

   - Characterize the solutions of associated conjugate nets and multiquadrilateral lattices
   - Dispersionless limit of these solutions, if any, and corresponding solutions of the Whitham hierarchy.

2. Study of the Riemann–Hilbert problem for multiple orthogonal polynomials and its relation with the aforementioned factorization problems in Lie groups. String equations characterizing these solutions.

3. Extended flows for the block Hankel reduced multicomponent Toda hierarchy.

5 Projected publications/articles resulting or to result from your grant

One or two papers regarding the results described in §3; i.e. one on the *Discrete flows and twistor equations for multicomponent Toda hierarchies* and another on *Dipersionfull Whitham hierarchy, factorization problems and dispersionless limit*.

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