Report MISGAM Exchange Grant 1658: Multicomponent Toda hierarchy: applications to matrix models

April 29, 2008

1 Purpose of the visit

The aim of the visit was to work further the appearance of the multicomponent KP hierachy in the description of Random Matrix Theory. In particular, we are interested in the results detailed in the preprint [1], in where solutions of the multicomponent KP hierarchy were built in terms of multiple orthogonal polynomials. Then, these constructions were applied to a number of relevant examples: biorthogonal polynomials, orthogonal polynomials, orthogonal polynomials in the circle and to nonintersecting Brownian motions. Multiple orthogonal polynomials were introduced in [2]. Later on, in [3] the use of the Rieman–Hilbert problem on the real line was presented to extend the mentioned results. This Riemann–Hilbert problem is a vast extension of the results of [4].

2 Description of the work carried out during the visit

The work during the visit may be divided into different parts. In the one hand, we will refer to the one regarding the discussions with Professor van Moerbeke on his work on matrix models and integrable systems and in particular in [1], and the discussions on the different descriptions of the multicomponent Toda hierarchy: factorization problem, Lax equations, Zakharov–

Shabat equations and twistor or string equations. We also discussed further on the recently discovered role of the multicomponent KP theory as a dispersionfull Whitham hierarchy. These results of Takasaki and Takebe, see [5], deserve further discussions.

On the other hand, there is the work on multicomponent integrable systems, extending the results of the papers [6] as we will describe in the following section. We also worked out the Takasaki-Takebe finding of the dispersionfull Whitham hierarchy within the multicomponent Toda scheme. In particular we study who the factorization problem described in [7] for the Whitham hierarchy may appear as a dispersionless limit of some other factorization problem at the dispersionfull level.

3 Description of the main results obtained

For the multicomponent Toda hierarchy we have the following results

1. Introduction of new discrete flows
2. Systematic discussion of the role of the Orlov operator
3. Formulation of twistor or string equations
4. Equivalence of the following descriptions of the hierarchy
   - Factorization problem
   - Lax equations
   - Zakharov–Shabat equations
   - Twistor or string equations
5. Introduction of additional symmetries of continuous and discrete type and corresponding string equations

For the dispersionfull Whitham hierarchy

- Description within the factorization problem formulation
- Finding of a good candidate to give the factorization description in the dispersionless limit
- Definition of good candidares for the Orlov operators going to Orlov operators in the dispersionless limit

4 Future collaboration with host institution

A number of problems subject to collaboration remain open

1. Analysis of the role of the multicomponent KP solutions of [1]:
   - Characterize the solutions of associated conjugate nets and multiquadrilateral lattices
   - Dispersionless limit of these solutions, if any, and corresponding solutions of the Whitham hierarchy.
2. Study of the Riemann–Hilbert problem for multiple orthogonal polynomials [3] and its relation with the aforementioned factorization problems in Lie groups. String equations characterizing these solutions.

5 Projected publications/articles resulting or to result from your grant

One or two papers regarding the results described in §3; i.e. one on the *Discrete flows and twistor equations for multicomponent Toda hierarchies* and another on *Dipersionfull Whitham hierarchy, factorization problems and dispersionless limit.*

References

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