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My visit to the SISSA Institute took place for one month, from 14 October to 14 November 2007. The host laboratory was the Sector of Mathematical Physics of SISSA (Head – Prof.B.Dubrovin).

The visit has been aimed to study a critical behavior near the point of "gradient catastrophe" of the solution to the Cauchy problem for the focusing nonlinear Schrödinger equation

$$i\epsilon\,\psi_t + \frac{\epsilon^2}{2}\psi_{xx} + |\psi|^2\psi = 0$$

with analytic initial data of the form $\psi(x,0;\epsilon) = A(x)\,e^{\frac{i}{\epsilon}S(x)}$. Namely, as it was proved in [1], the leading term of the asymptotics is described by the *tritronquée* solution to the Painlevé-I equation. It is defined as the solution of the ODE

$$u_{\zeta\zeta} = 6u^2 - \zeta,$$

which does not have poles in the sector $|\arg\zeta| < 4\pi/5$ for sufficiently large $|\zeta|$. This solution has been distinguished long ago by P.Boutroux [2] in connection with analytic theory of nonlinear ODEs in the complex plane.

The authors of [1] posed a conjecture that *tritronquée* solution to the Painlevé-I equation has *no* poles in the sector $|\arg\zeta| < 4\pi/5$. This was confirmed both by numerical modelling and by the study of nonlinear Schrödinger approximations. During my visit, we have found new confirmations of this conjecture and outlined a way of a strict proof.

First, an effective algorithm for the Padé approximation has been written in "Mathematica" symbolic package. It constructs the main diagonal Padé rational approximation for any solution of Painlevé - I equation, defined by the initial conditions at the origin. The initial conditions for the *tritronquée* solution, found previously in [3], have been checked and improved up to 64 digit accuracy. Then, apllying the Padé approximation algorithm, the coordinates of poles were found as shown in Figure 1. The picture shows the level lines $|u(\zeta)| = 3$ with the help of `ContourPlot` procedure of "Mathematica". Green lines correspond to the rays $|\arg\zeta| = 2\pi n/5$, $n = 1,\ldots,5$. All the poles are situated in sector $4\pi/5 < |\arg\zeta| < 6\pi/5$, which confirms the conjecture of the paper [1].

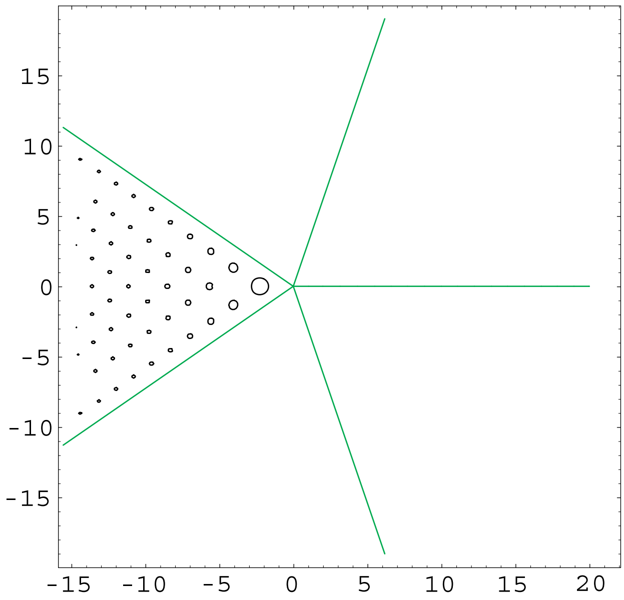

Figure 1: Level lines of the *tritronquée* solution $|u(\zeta)|$ on the complex plane ζ near the origin.

Second, the Padé approximation has confirmed the position of the nearest to the origin pole as $\zeta_0 = -2.3841687...$, which is in line with the result of [3]. On the other hand, the asymptotic distribution of poles at infinity found in [4], seems to be in good approximation with Padé approximation, starting from $|\zeta| \geq 10$.

Third, we found a way to prove the conjecture stated in [1]. The idea of proof is based on the Isomonodromy Deformation Method, described in [5]. In this method, the *tritronquée* solution is fixed by the monodromy data of an auxiliary linear system of ODEs. Assuming ζ to be a pole of Ω in forbidden sector $|\arg\zeta| < 6\pi/5$, we come to a standard Sturm-Lioville problem on the real axis for the second order ODE with polynomial potential. Then, if this problem has no solution, so is the inverse monodromy problem for $\Omega(\zeta)$. This contradiction will prove the non-existence of poles in the forbidden sector. The similar construction for the case of Painlevé - II equation has been studied in Chapter 10 of the book [5].

We plan to write a joint paper, describing the above results, with prof. B.Dubrovin and Dr. T.Grava. I wold like to thank them for helpful discussions and hospitality during my visit to SISSA.

References

[1] B.Dubrovin, T.Grava and C.Klein, On universality of critical behaviour in the focusing nonlinear Schrödinger equation, elliptic umbilic catastrophe and the *tritronquée* solution to the Painlevé-I equation, arXiv:0704.0501v2, (2007).

[2] P.Boutroux, Recherches sur les transcendants de M. Painlevé et l'étude asymptotique des équations différentielles du second ordre. *Ann. École Norm* **30** (1913) 265 - 375.

[3] N.Joshi, A.Kitaev, On Boutroux's tritronquée solutions of the first Painlevé equation. *Stud. Appl. Math.* **107** (2001) 253–291.

[4] A.Kapaev, Quasi-linear Stokes phenomenon for the Painlevé first equation. *J. Phys. A: Math. Gen.* **37** (2004) 11149–11167.

[5] A.R.Its, V.Yu.Novokshenov, The isomonodromic deformation method in the theory of Painlevé equations, Lect.Notes in Math., v.1191 (Springer, Berlin, 1986)