Short description of the proposed project

Nonlocal Poisson structures for the Camassa-Holm hierarchy

The concept of bi-(or more generally poly-)Hamiltonian or Poisson structures is known, since the beautiful observation of F. Magri, to play the role of one of the most important and useful methods in study of Integrable Systems. A pair of compatible (or coordinated) Poisson structures leads to so-called "Recursion Operator" which can be used to produce an infinity of new "superior" Poisson structures which are basically non-local. An (arche)typic example of this situation in the theory of Soliton Equations gives the pair of Gel'fand-Dikii (GD) Poisson structures associated with any semisimple Lie algebra on the KdV hierarchy phase space (Adler-Gelfand-Dikii-Lenard-Magri).

For the case of sl_2-KdV hierarchy all "higher" GD brackets containing non-local terms and a generating function for all this structures are described explicitly in terms of the Baker-Akhiezer functions of the adjoint linear problem by B. Enriquez, A. Orlov and V. Rubtsov.

The main goal of the project is to extend a list of integrable systems admitting so called weakly non-local Poisson (or Hamiltonian) structures. The general definition and study of this important class of Poisson structures arising within some Soliton Integrable Systems were proposed recently by A. Maltsev and S. Novikov. Their description coveres the NLS (Nonlinear Schroedinger) hierarchy case.

It is well-known now that there are several different non-linear PDE's hierarchies on the same phase space with KdV. Among them there are the Camassa-Holm (CH), the Hunter-Saxton (HS) and the Harry-Dym (HD)equations which are also bi-hamiltonian and admit infinite number of local and non-local conservation laws (M. Pedroni, P. Casatti, G. Ortenzi et all.).

We are constructing a generating series for the Camassa-Holm (extended) bihamiltonian phase space using our previuos results.

Two compatible Poisson structures for KdV phase spaces (referring usually as GD-brackets) are defined as follows: the first ("GD1") has the form $\{u(x), u(y)\} = \partial_x \delta_{xy}$ when $u(x)$ is a rapidly decreasing on the real line function).

The second ("GD2" or, more correctly, Lenard-Magri) bracket has the form

$$\{u(x), u(y)\} = \frac{1}{4}\delta^{'''}(x-y)\frac{1}{2}(u(x)+u(y))\delta^{'}(x-y)$$

In this case, the induced Hamiltonian map is invertible and the inversion gives birth to "integral" terms in the recursion operator R expression and the "higher" GD Poisson structures are nonlocal, i.e. they may contain the terms of the type $u(x)v(y)\partial_x^{-1}\delta(x-y)$. All this structures are encoded in the generating function for the "higher" GD structures and the non-local part of the $n-$th Poisson bracket has the following form:

$$\{u(x), u(y)\}_n = \sum_{i=1}^{n-2} K_i^{'}(x)K_{n-1-i}^{'}(y)\partial_x^{-1}\delta(x-y) + \text{local terms}, \quad (1)$$

where K_i are polynomials in u and its derivatives, such that the higher KdV flows express by $\partial_i u = \partial_x K_i$.

The beauty of (1) (and a difficulty to prove it) lies in the fact that, in spite of multiple

”integrations”, the nonlocal terms rest always in the form

$$\sum_i u_i(x)v_i(y)\partial_x^{-1}\delta(x-y)$$

with differential polynomials u_i and v_i in u.

We want to establish a similar description to all “higher” (non-local) Poisson structures for the other hierarchies.

We have constructed an analogue of the generating series for these “higher” Poisson structures for CH during the visit of G. Ortenzi and M. Pedroni in Angers in November 2006.

The aim of my short visit to the University of Milan -Bicocca is to finish our paper which we have started in November 2006. I want to spend a week in Milan in June 2007 (arrival date: june 10- departure june 16). My host will be Prof. Franco Magri and Dr. Giovanni Ortenzi.