1.1 A short description of the proposed project work (about 1000 words)

Title: "On the notions of duality in finite and infinite-dimensional bi-Hamiltonian integrable systems"

The duality for finite dimensional integrable systems on a cotangent space is a notion introduced in the paper [5], co-authored by the hosting professor. Let us consider an integrable system on a cotangent space and its action-angle variables. There are two natural foliations of its phase space: The Lagrangian foliation which is given fixing the action variables and the canonical foliation obtained by canonical projection. Two integrable systems on a cotangent space are called *dual* if it exists a symplectic involution σ mapping the Lagrangian foliation of the first system on the canonical foliation of the second one and viceversa. The three prototypic pairs of dual integrable systems are the trigonometric Calogero-Moser with the rational Ruijsenaars-Schneider, the trigonometric Ruijsenaars-Schneider with itself and the rational Calogero-Moser with itself.
In the paper [1] we frame the previous notion of duality, for the rational Calogero-Moser system, in the context of bi-Hamiltonian systems and we also give a slight generalization of the original definition of duality between integrable systems on cotangent spaces.
The starting point of our construction are the papers [9, 11], where the authors study a reduction from the cotangent space of the Lie algebra $\mathfrak{gl}_n$ onto the phase space of the rational Calogero-Moser system. We remark that the space $T^*\mathfrak{gl}_n \simeq \mathfrak{gl}_n \times \mathfrak{gl}_n$ is bi-Hamiltonian with respect to the inverse of the canonical symplectic form ω and the Poisson tensor

$$P_1 = \begin{pmatrix} 0 & X\cdot \\ \cdot X & [Y,\cdot] \end{pmatrix} \qquad (X,Y) \in T^*\mathfrak{gl}_n.$$

This Poisson structure is obtained from the recursion operator N which is the *complete lift* (see e.g. [2]) of the torsionless $(1,1)$ tensor $L_X : V \to XV$, $X \in \mathfrak{gl}_n$, $Y \in T^*_X\mathfrak{gl}_n \simeq \mathfrak{gl}_n$. The simultaneous conjugation of the group GL_n on a suitable chosen subset $M \subset T^*\mathfrak{gl}_n$ leaves invariant the tensors P_0 and P_1. The functions $F^{ij} = tr(X^iY^j)$ are invariant with respect to this group action and are suitable co-ordinates on the space M/GL_n. The phase space and the bi-Hamiltonian structure of the rational Calogero-Moser system are obtained by a projection along some F^{ij}. However, on the subset M, there exists another bi-Hamiltonian pair obtained from P_0 and P_1 simply exchanging X and Y. After a similar reduction process one obtain a hierarchy which is again the rational Calogero-Moser system. The existence of two different bi-Hamiltonian pencils on $T^*\mathfrak{gl}_n$ whose reduction leads to the same system is, from our point of view, the motivation of the self-duality. A still open problem is to extend this construction to the other pairs of dual systems.

A totally different picture appears in the context of infinite-dimensional systems. In the paper [6] the authors introduce a notion of duality for infinite dimensional hierarchies: Two bi-Hamiltonian systems are called *tri-Hamiltonian dual* if they share the same Poisson manifold and the Poisson structures are obtained one from the other by a shift of a part of the structures themselves. If a system is bi-Hamiltonian with respect to the pair of structures $P_0 = P_c + P_a$ and $P_1 = P_b$, its dual will be bi-Hamiltonian with respect to $P_0^D = P_a$ and $P_1^D = P_c + P_b$. It is worth to remark that this shift it is not possible in general, but it requires that P_a, P_b, and P_c are compatible, i.e. every linear combination of them must be a Poisson tensor. This connection between the Poisson structures gives also a correspondence between the hierarchies and the related conserved quantities. Actually the standard bi-Hamiltonian theory states that there is a natural way to associate a hierarchy to a bi-Hamiltonian manifold if the Poisson pencil admits a Casimir. The most celebrated example of this mapping between integrable hierarchies is the Camassa-Holm/Kortweg-de Vries duality. The Poisson pencil of the two systems has the structure

$$P_{\alpha,\beta,\gamma} = \alpha\partial_x + \beta\partial_x^3 + \gamma(u\partial_x + \partial_x u) \tag{1}$$

and fixing the coefficients as

$$\begin{aligned} P_{KdV,\lambda} &= P_{-\lambda,\frac{1}{4},\frac{1}{2}} := -\lambda\partial_x + \frac{1}{4}\partial_x^3 + \frac{1}{2}(u\partial_x + \partial_x u) \\ P_{CH,\lambda} &= P_{-\lambda,\frac{\lambda}{4},\frac{1}{2}} := -\lambda\left(\partial_x - \frac{1}{4}\partial_x^3\right) + \frac{1}{2}(u\partial_x + \partial_x u), \end{aligned} \tag{2}$$

where λ is the spectral parameter, we obtain the two pencils. The different choice of the coefficients α, β, γ corresponds to the shift of the tensor ∂_x^3 from the first to the second structure. The generators of the Hamiltonian densities of the two dual hierarchies satisfy the Riccati equation $-2\beta(h_x + h^2) = \gamma u + \frac{\alpha}{2}$ where the coefficients are the same given in (2). With Prof. Roubtsov and Prof. Pedroni we have studied some properties of the CH/KdV duality: In [10] the duality is used, extending the results present in [3], in order to characterize the nonlocality of the higher Poisson structures in the CH case which is related to the inverse of the inertia tensor of the equations introduced in [7].
The general properties of this kind of duality are not well understood, e.g. the fact that it connect, in a large number of cases, smooth soliton and non-smooth soliton equations [6, 8, 4] or its relation with the reciprocal transformation between dual hierarchies.

1.2 Aim of the visit

The main aim of my visit is to continue the study of dual systems in the infinite-dimensional case [6] and the properties of the related higher Poisson structures in dispersive multicomponent cases. The final goal is to understand if there is an analogue in the infinite dimensional case of the involution σ defined for the finite dimensional systems.
A second aim of the visit is to generalize the construction presented in [1] to the other known cases of finite-dimensional duality cited above or for different Lie algebras. This result is interesting in order to understand if it is possible to re-state the definition of duality in the framework of the bi-Hamiltonian context, which is a useful way to understand a possible relation with the infinite-dimensional case.

References

[1] C. Bartocci, G. Falqui, I. Mencattini, G. Ortenzi, M. Pedroni, *On the geometric origin of the bi-Hamiltonian structure of the Calogero-Moser system*, work in progress

[2] C. Bartocci, G. Falqui, M. Pedroni, *A geometric approach to the separability of the Neumann-Rosochatius system*, Differential Geom. Appl. 21 (2004), no. 3, 349–360.

[3] B. Enriquez, A. Orlov, V. Rubtsov, *Higher Hamiltonian structures (the $\mathfrak{sl}_2$ case)*, JETP Letters, Volume 58, Issue 8, October 25, 1993, pp.658-664.

[4] G. Falqui *On a Camassa-Holm type equation with two dependent variables*, J. Phys. A: Math. Gen., 39 (2006), 327–342.

[5] V. Fock, A. Gorsky, N. Nekrasov, V. Rubtsov, *Duality in integrable systems and gauge theories*, J. High Energy Phys. 2000, no. 7, Paper 28, 40 pp.

[6] P. J. Olver, P. Rosenau, *Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support*, Phys. Rev. E (3) 53 (1996), no. 2, 1900–1906.

[7] B. Khesin, G. Misiolek, *Euler equations on homogeneous spaces and Virasoro orbits*, Advances in Math. 176 (2003), 116–144.

[8] M. Chen, S.-Q.Liu , Y. Zhang, *A two-component generalization of the Camassa-Holm equation and its solutions*, Lett. Math. Phys. 75 (2006), no. 1, 1–15.

[9] D. Kazhdan, B. Kostant, S. Sternberg, *Hamiltonian group actions and dynamical systems of Calogero type*, Comm. Pure Appl. Math. 31 (1978), no. 4, 481–507.

[10] G. Ortenzi, M. Pedroni, V. Rubtsov *On the higher Poisson structures of the Camassa-Holm hierarchy*, preprint of the Department of Mathematics of the University of Angers, submitted for publication.

[11] G. Wilson, *Collisions of Calogero-Moser particles and an adelic Grassmannian*, with an appendix by I. G. Macdonald. Invent. Math. 133 (1998), no. 1, 1–41.