Short description of the proposed project work and the aim of the visit

Consider unitary random matrix ensembles on the space of $n \times n$ Hermitian matrices with a probability density of the form

$$\frac{1}{Z_n} e^{-n \operatorname{Tr} V(M)} dM, \tag{1}$$

where V is a real analytic function satisfying some growth condition at infinity. The factor n in the exponent is a simple re-scaling of the eigenvalues, such that the limiting mean density of eigenvalues exists and is supported on a finite union of intervals. Useful information about the statistics of the eigenvalues is contained in the so-called two-point kernel

$$K_n(x,y) = e^{-\frac{n}{2}V(x)} e^{-\frac{n}{2}V(y)} \sum_{k=0}^{n-1} p_{k,n}(x) p_{k,n}(y), \tag{2}$$

where $p_{k,n}$ denotes the k-th degree orthogonal polynomial with respect to the weight $e^{-nV(x)}$.

It turns out that local scaling limits of the two-point kernel are universal: the limiting kernel should only depend on the scaling regime. The two regular scaling regimes are the following,

- bulk scaling, near points where the limiting mean eigenvalue density is positive,
- edge scaling, near edge points of the spectrum.

In the bulk case the limiting kernel is the sine kernel,

$$K^{bulk}(u,v) = \frac{\sin \pi(u-v)}{\pi(u-v)}. \tag{3}$$

In the edge case the Airy kernel appears in the large n-limit,

$$K^{edge}(u,v) = \frac{\operatorname{Ai}(x) \operatorname{Ai}'(y) - \operatorname{Ai}(y) \operatorname{Ai}'(x)}{x-y}. \tag{4}$$

Near singular points, three other scaling regimes can occur. The singular points are classified as follows,

- type I singular points, which are singular points outside the support of the limiting mean eigenvalue density,

- type II singular points, which are points in the interior of the spectrum where the limiting mean eigenvalue density vanishes,
- type III singular points, which are edge points of the spectrum where the limiting mean eigenvalue density vanishes at a higher order than in the regular case.

The case of a type II singular point has already been studied and leads to a limiting kernel built out of functions associated with the Hastings-McLeod solution of the Painlevé II equation. In a double scaling limit this describes the closing of a gap between two intervals. Type I singular points, which describe the opening of a new band, have been studied in the physics literature [6] but have not been treated rigorously yet. It might be an interesting problem to find the limiting kernel in this case, in a double scaling limit where a new interval of the spectrum opens. Type III singular points have also been studied [2] and here the limiting kernel can be given in terms of a fourth order analogue of the Painlevé I equation. These universality results can be obtained by using Riemann-Hilbert problems, in particular by applying the Deift/Zhou steepest-descent method on the Riemann-Hilbert problem for orthogonal polynomials [3]. The construction of local parametrices near the singular points and the edge points is crucial here. In the double scaling limits for the type II and type III singular points, asymptotics for the recurrence coefficients of the related orthogonal polynomials have also been computed. For the type III case, those recurrence coefficients have the same form as the one appearing in a conjecture of Dubrovin (who is at SISSA) [5]. He conjectured asymptotics for solutions of Hamiltonian perturbations of hyperbolic equations.

Hyperbolic equations of the form

$$u_t + a(u)u_x = 0 \tag{5}$$

can be perturbed to a Hamiltonian equation of the form

$$u_t + a(u)u_x + \epsilon\left[b_1(u)u_{xx} + b_2(u)u_x^2\right] + \epsilon^2\left[b_3(u)u_{xxx} + b_4(u)u_xu_{xx} + b_5(u)u_x^3\right] + \cdots = 0, \tag{6}$$

where ϵ is small and $b_1, b_2, \ldots$ are smooth functions. These equations have been studied by Dubrovin in [4, 5], where he formulated the universality conjecture about the behavior of a generic solution to a general perturbed Hamiltonian equation (6) near the point (x_0, t_0) of *gradient catastrophe* of the unperturbed solution (5). He argued that this behavior is described by a special solution of the fourth order analogue of the Painlevé I equation. He also conjectured that this special solution has no poles on the real line, which was proven in [1].

In [7], Grava (who is also at SISSA) and Klein did numerical calculations for the particular example (of a perturbed Hamiltonian equation) of the small dispersion limit of the KdV equation,

$$u_t + 6uu_x + \epsilon^2 u_{xxx} = 0, \qquad \text{with initial condition} \qquad u(x,0) = u_0(x).$$

Before the time of gradient catastrophe t_0, solutions turn out to behave nicely. When approaching the critical time t_0, the slope of the function blows up near x_0 and fast oscillations near x_0 set in. The transition near the critical point x_0 should be described in terms of the real pole-free solution of the fourth order analogue of the Painlevé I equation. Furthermore at the left edge of the oscillatory interval, the transition is expected to be described in terms of the Hastings-McLeod solution of the Painlevé II equation. At the right edge of the oscillatory interval, it is not known yet what kind of behavior one can expect. Possibly some of these problems can be tackled using Riemann-Hilbert problems and similar methods as the ones used for random matrix ensembles.

I think the SISSA institute would be an excellent environment for me to study some of the above mentioned problems.

References

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