Proposed Project Work for Short Visit Grants

— Blow-up analysis for the Camassa-Holm equation

We consider the Cauchy problem for the Camassa-Holm equation in $\mathbb{R}$

$$\begin{cases} u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, & t > 0,\ x \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

This equation, models wave motion in shallow water region with u denoting the height of the water above a flat bottom.

Equation (1) is a integrable system, so there are infinite many conservation laws associated to it. However, unlike the KdV, one of the most significant phenomenon of the Camassa-Holm equation is wave breaking. In this case, the strong (smooth) solution $u(x,t)$ itself remains bounded but its first order derivative $u_x(x,t)$ becomes infinity as (x,t) goes to some point (x_0, t_0). Various conditions (see the papers [2,3,5,14] for example) has been established on the initial datum $u_0(x)$ to guarantee finite time singularity formation (wave breaking) for the corresponding strong solution. In particular, H. McKean [1] (see [2] for a simple proof) proved that the solution to (1) breaks down if and only if some portion of the positive part of $y_0(x) = (1-\partial_x^2)u_0(x)$ lies to the left of some portion of its negative part.

One of the goals of the visit is to cooperate with Professor T. Ratiu at EPFL to make deeper understanding on the evolution of the corresponding solution with McKean's initial condition. We will try to give an alternative simple proof to get more information of the solution as it near the blow-up time.

Although the necessary and sufficient condition is established, there is no understanding for the profile of the corresponding solution at the blow-up time. Up to now, it is only known that the solution is quite different from the shock profile when it blows up. The other goal is to work with Professor T. Ratiu to do some blow-up analysis for the corresponding solution.

References

[1] McKean, H. P., Breakdown of a shallow water equation, Asian J. Math., 2 (1998), No.4, 867–874.

[2] McKean, H. P., Breakdown of the Camassa-Holm equation. Comm. Pure Appl. Math. 57 (2004), no. 3, 416–418.