Final report on Short Visit Grant within ESF Scientific Programme MISGAM

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Explicit solutions to Knizhnik-Zamolodchikov equations and representation theory

Purpose of the visit

The aim of the visit (29.03 - 14.04. 2006) by Prof A.P. Veselov (Loughborough University, UK) to ETH, Zurich was to continue collaboration between the visitor and the host Prof G. Felder on the Knizhnik-Zamolodchikov equations in relation with the representation theory.

Description of the work carried out during the visit

Let G be a finite Coxeter group, R be the corresponding root system, m_α, $\alpha \in R$ be a system of multiplicities, which is a G-invariant function on R. Let M be an irreducible representation of G and define the *Knizhnik–Zamolodchikov equation* related to M as the following system

$$\partial_\xi \psi = \sum_{\alpha \in R_+} m_\alpha \frac{(\alpha, \xi)}{(\alpha, z)} (s_\alpha + 1)\psi,$$

where s_α are the corresponding reflections acting on M-valued functions $\psi(x)$. If the multiplicities m_α are integers then all the solutions of the corresponding systems are polynomial (see [1]). The finding of these solutions is actually an important part of the description of the so-called m-harmonic polynomials [2]. In the paper [1] the corresponding solutions were found explicitly in the simplest case of the reflection representation of $G = S_N$.

During the visit we have been working on the case of the general representation of S_N trying to find the effective formulas for the solutions of the corresponding KZ equations.

Description of the main results obtained during the visit

Our main result is an explicit integral formula for the solutions of the KZ equation

$$\partial_i \psi = m \sum_{j \neq i}^{N} \frac{s_{ij}+1}{z_i - z_j} \psi, \quad i = 1, \ldots, N$$

with values in an arbitrary irreducible representation of the symmetric group S_N for any integer m. The approach is based on the results of Matsuo [3] and Schur-Weyl duality.

The main idea is following. Let V be an n-dimensional complex vector space. Then the symmetric group S_N on N letters acts on the N-fold tensor product $V^{\otimes N} = V \otimes \cdots \otimes V$ by permutations of factors and this action commutes with the diagonal action of $GL(V)$. The classical Schur–Weyl theorem states that, as a $GL(V) \times S_N$ module, $V^{\otimes N}$ has a decomposition into a direct sum

$$V^{\otimes N} \simeq \oplus_\lambda M^\lambda \otimes W^\lambda$$

where M^λ are inequivalent irreducible $GL(V)$-modules and W^λ are inequivalent irreducible S_N-modules. The sum is over partitions of N into at most n parts, namely sequences of integers $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ with $\sum \lambda_i = N$ (or equivalently, *Young diagrams* with N boxes and at most n rows). Moreover if $n \geq N$ all irreducible S_N modules appear. Thus we can realise every irreducible S_N-module as $W^\lambda = \mathrm{Hom_{GL(V)}(M^\lambda, V^{\otimes N})}$, for any V of dimension $\geq N$. There is a standard construction of a basis v_T of W_λ labeled by the so-called *standard tableaux* T on λ (see [4, 5]).

The basis of solutions of the KZ equation with values in W^λ can be also labeled by the set of standard tableaux $\mathcal{T}(\lambda)$:

$$\psi_T(z_1, \ldots, z_N) = \sum_{T' \in \mathcal{T}(\lambda)} \psi_{T,T'}(z_1, \ldots, z_N) v_{T'}.$$

The components $\psi_{T,T'}(z_1, \ldots, z_N)$ are known to be polynomial in $z_1, \ldots, z_N$ [1]. We found an explicit formula for $\psi_{T,T'}(z_1, \ldots, z_N)$ as an integral

$$\psi_{T,T'} = \int_{\sigma_T} \omega_{T'}$$

of some rational differential form $\omega_{T'}$ over certain cycle σ_T in the top homology of the following configuration space $C_\lambda(z_1, \ldots, z_N)$ related to Young diagram λ.

Let $\lambda = (\lambda_1, \ldots, \lambda_n) = (m_0 - m_1, m_1 - m_2, \ldots, m_{n-2} - m_{n-1}, m_{n-1})$, so

$$m_0 = \lambda_1 + \lambda_2 + \ldots + \lambda_n = N, \; m_1 = \lambda_2 + \ldots + \lambda_n, \; \ldots, m_{n-2} = \lambda_{n-1} + \lambda_n, m_{n-1} = \lambda_n.$$

Consider n finite sets $X_0, X_1, \ldots, X_{n-1}$ of points on the complex plane $\mathbf{C}$ consisting of $m_0 = N, \ldots, m_{n-1}$ points respectively with the condition that X_i and X_{i+1} have no common points for all $i = 1, \ldots, n-1$. Let us denote the elements of X_0 as $z_1, \ldots, z_N$ and fix them. The corresponding configuration space of all admissible $\{X_1, \ldots, X_{n-1}\}$ is our $C_\lambda(z_1, \ldots, z_N)$.

On this space we have a natural action of the group $G_\lambda = S_{m_1} \times S_{m_2} \times \ldots \times S_{m_{n-1}}$. One can show that the skew-symmetric part of the top homology group $H_{top}(C_\lambda(z_1, \ldots, z_N), \mathbf{C})$ can be naturally identified with the weight space $(V^{\otimes N})_\lambda$. This defines the cycles $\sigma_T \in H_{top}(C_\lambda(z_1, \ldots, z_N))$. The differential forms ω_T on $C_\lambda(z_1, \ldots, z_N)$ have explicit rational expressions derived from Matsuo's paper [3], and the integral $\int_{\sigma_T} \omega_{T'}$ can be effectively computed as an iterated resudue.

The details and proofs are currently under preparation for publication [6]

Conclusion

The visit was very successful. We have managed to solve the problem we had in mind. We have also discussed some problems of common interest, which we hope to approach in the future.

References

[1] G. Felder, A.P. Veselov *Action of Coxeter groups on m-harmonic polynomials and KZ equations.* Moscow Math Journal, 2003, vol.3, no.4, 1269-1291.

[2] M. Feigin and A.P. Veselov *Quasi-invariants of Coxeter groups and m-harmonic polynomials.* Int. Math. Res. Notices (2002), no. 10, 521-545.

[3] A. Matsuo *An application of Aomoto-Gelfand hypergeometric functions to the SU(n) Knizhnik-Zamolodchikov equation.* Commun. Math. Phys. 134 (1990), 65-77.

[4] William Fulton. Young Tableaux, with Applications to Representation Theory and Geometry. Cambridge University Press, 1997

[5] R. Goodman and N. R. Wallach. Representations and Invariants of the Classical Groups. Cambridge University Press, 1998

[6] G. Felder, A.P. Veselov *Polynomial solutions of the KZ equations and Schur-Weyl duality.* In preparation.