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Abstract. There is developed a symplectic approach for partial solving the problem of algebraic-analytical construction of integral submanifold imbeddings for integrable via the abelian and nonabelian Liouville-Arnold theorems Hamiltonian systems on canonically symplectic phase spaces. The fundamental role of so called Picard-Fuchs type equations is revealed and their differential-geometric and algebraic properties are studied in detail. Some interesting examples of integrable Hamiltonian systems demonstrating the algorithm of investigating the integral submanifolds imbedding mapping are studied in detail. The canonical reduction method on canonically symplectic manifolds is analized in detail, the relationships with the geometric properties of associated principal fiber bundles endowed with connection structures are stated. Some results devoted to studying geometrical properties of nonabelian Yang-Mills type gauge field quations are presented. The differential-geometric and topological structure of Delsarte transmutation operators and associated with them Gelfand-Levitan-Marchenko type eqautions are studied making use of the De Rham-Hodge-Skrypnik differential complex. The relationships with spectral theory and special Berezansky type congruence properties of Delsarte transmuted operators are stated. Some applications to multidimensional differential operators are done including three-dimensional Laplace operator, two-dimensional classical Dirac operator and its multidimensional affine extension, related with self-dual Yang-Mills eqautions. The soliton like solutions to the related set of nonlinear dynamical systems are discussed.

CONTENTS

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1. The differential geometric analysis of the integral manifold imbedding mapping problem: the Liouville-Arnold integrability by quadratures, Picard-Fuchs type equations and Hamilton-Jacobi separation of variables

Introduction As is well known [1,4,DNF], the integrability by quadratures of a differential equation in space $\mathbb{R}^n$ is a method of seeking its solutions by means of finite number of algebraic operations (together with inversion of functions) and "quadratures"- calculations of integrals of known functions.

Assume that our differential equation is given as a Hamiltonian dynamical system on some appropriate symplectic manifold $(M^{2n},\omega^{(2)})$, $n \in \mathbb{Z}_+$, in the form

$$du/dt = \{H, u\}, \tag{1.1}$$

where $u \in M^{2n}$, $H : M^{2n} \to \mathbb{R}$ is a sufficiently smooth Hamiltonian function [1,4,DNF] with respect to the Poisson bracket $\{\cdot,\cdot\}$ on $\mathcal{D}(M^{2n})$, dual to the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$, and $t \in \mathbb{R}$ is the evolution parameter.

More than one hundred and fifty years ago French mathematicians and physicists, first E. Bour and next J. Liouville, proved the first "integrability by quadratures" theorem which in modern terms [33] can be formulated as follows.

Theorem 1.1. *Let $M^{2n} \simeq T^*(\mathbb{R}^n)$ be a canonically symplectic phase space and there be given a dynamical system(1.1) with a Hamiltonian function H: $M^{2n}\times\mathbb{R}_t \to \mathbb{R}$, possessing a Poissonian Lie algebra $\mathcal{G}$ of $n\in \mathbb{Z}_+$ invariants $H_j : M^{2n}\times\mathbb{R}_t \to\mathbb{R}$, $j=\overline{1,n}$, such that*

$$\{H_i, H_j\} = \sum_{s=1}^{n} c_{ij}^s H_s, \tag{1.2}$$

and for all i,j,k $=\overline{1,n}$ the $c_{ij}^s \in\mathbb{R}$ are constants on $M^{2n}\times\mathbb{R}_t$. Suppose further that

$$M_h^{n+1} =:= \{(u,t) \in M \times \mathbb{R}_t : h(H_j) = h_j,\ \ j = \overline{1,n},\ h \in \mathcal{G}^*\}, \tag{1.3}$$

the integral submanifold of the set $\mathcal{G}$ of invariants at a regular element $h\in \mathcal{G}^*$, is a well defined connected submanifold of $M\times\mathbb{R}_t$. Then, if :

i) all functions of $\mathcal{G}$ are functionally independent on M_h^{n+1};

ii) $\sum_{s=1}^{n} c_{ij}^s h_s = 0$ for all $i,j =\overline{1,n}$;

iii) the Lie algebra $\mathcal{G} = span_{\mathbb{R}}\ \{H_j : M^{2n}\times\mathbb{R}_t \to\mathbb{R}\text{: } j\overline{= 1,n}\}$ is solvable, the Hamiltonian system (1.1) on M^{2n} is integrable by quadratures.

As a simple corollary of the Bour-Liouville theorem one gets the following:

Corollary 1.2. *If a Hamiltonian system on $M^{2n} =T^*(\mathbb{R}^n)$ possesses just $n\in \mathbb{Z}_+$ functionally independent invariants in involution, that is a Lie algebra $\mathcal{G}$ is abelian, then it is integrable by quadratures.*

In the autonomous case when a Hamiltonian $H = H_1$, and invariants $H_j : M^{2n} \to \mathbb{R}$, $j = \overline{1,n}$, are independent of the evolution parameter $t \in\mathbb{R}$, the involutivity condition $\{H_i, H_j\} = 0$, $i,j = \overline{1,n}$, can be replaced by the weaker one $\{H, H_j\} = c_j H$ for some constants $c_j \in \mathbb{R}$, $j = \overline{1,n}$.

The first proof of Theorem 1.1. was based on a result of S. Lie, which can be formulated as follows.

Theorem 1.3 *(S. Lie) Let vector fields $K_j \in \Gamma(M^{2n})$, $j = \overline{1,n}$, be independent in some open neighborhood $U_h \in M^{2n}$, generate a solvable Lie algebra $\mathcal{G}$ with* *respect to the usual commutator* $[\cdot,\cdot]$ *on* $\Gamma(M^{2n})$ *and* $[K_j, K] = c_j K$ *for all* $j=\overline{1,n}$, *where* $c_j \in\mathbb{R}$, $j=\overline{1,n}$, *are constants. Then the dynamical system*

$$du/dt = K(u), \tag{1.4}$$

where $u \in U_h \subset M^{2n}$, *is integrable by quadratures.*

Example 1.4 *Motion of three particles on line* $\mathbb{R}$ *under uniform potential field.*

The motion of three particles on the axis $\mathbb{R}$ pairwise interacting via a uniform potential field $Q(\|\cdot\|)$ is described as a Hamiltonian system on the canonically symplectic phase space $M = T^*(\mathbb{R}^3)$ with the following Lie algebra $\mathcal{G}$ of invariants on M^{2n}:

$$H = H_1 = \sum_{j=1}^{3} p_j^2/2m_j + \sum_{i<j=1}^{3} Q(\|q_i - q_j\|), \tag{1.5}$$

$$H_2 = \sum_{j=1}^{3} q_j p_j, \qquad H_3 = \sum_{j=1}^{3} p_j, \tag{1.6}$$

where $(q_j, p_j) \in T^*(\mathbb{R})$, $j=\overline{1,3}$, are coordinates and momenta of particles on the axis $\mathbb{R}$. The commutation relations for the Lie algebra $\mathcal{G}$ are

$$\{H_1, H_3\} = 0, \qquad \{H_2, H_3\} = H_3, \quad \{H_1, H_2\} = 2H_1, \tag{1.7}$$

hence it clearly solvable. Taking a regular element $h \in \mathcal{G}^*$, such that $h(H_j) = h_j = 0$, for $j = 1$ and 3, and $h(H_2) = h_2 \in \mathbb{R}$ being arbitrary, one obtains the integrability of the problem above in quadratures.

1.1. **Abelian integrabilty by quadratures.** In 1974 V. Arnold proved [4] the following important result known as the commutative (abelian) Liouville-Arnold theorem. **Theorem 1.5** *(J.Liouville-V. Arnold). Suppose a set* $\mathcal{G}$ *of functions* $H_j : M^{2n} \to \mathbb{R}$, $j=\overline{1,n}$, *on a symplectic manifold* M^{2n} *is abelian, that is*

$$\{H_i, H_j\} = 0 \tag{1.8}$$

for all $i,j=\overline{1,n}$. If *on the compact and connected integral submanifold* $M_h^n=\{u\in M^{2n}: h(H_j)=h_j \in \mathbb{R},\ j=\overline{1,n},\ h\in \mathcal{G}^*\}$ *with* $h\in \mathcal{G}$ *being regular, all functions* $: M^{2n} \to \mathbb{R}$, $j=\overline{1,n}$, *are functionally independent, then* M_h^n is *diffeomorphic to the n-dimensional torus* $\mathbb{T}^n \simeq M^{2n}$, *and the motion on it with respect to the Hamiltonian* $H=H_1 \in \mathcal{G}$ *is a quasi-periodic function of the evolution parameter* $t\in \mathbb{R}$.

A dynamical system satisfying the hypotheses of Theorem 1.5 is called completely integrable.

1.2. **Non-abelianIn integrability by quadratures.** In 1978 Mishchenko and Fomenko [2] proved the following generalization of the Liouville-Arnold theorem 1.5:

Theorem 1.6. *(A. Mishchenko-A. Fomenko) Assume that on a symplectic manifold* $(M^{2n},\omega^{(2)})$ *there is a nonabelian Lie algebra* $\mathcal{G}$ *of invariants* $H_j : M \in \mathbb{R}$, $j=\overline{1,k}$, *with respect to the dual Poisson bracket on* M^{2n}, *that is*

$$\{H_i, H_j\} = \sum_{s=1}^{k} c_{ij}^s H_s, \tag{1.9}$$

where all values $c_{ij}^s \in \mathbb{R}$, $i,j,s =\overline{1,k}$, *are constants, and the following conditions are satisfied:*

i) the integral submanifold $M_h^r{:=}\{u{\in}M^{2n}{:}h(H_j){=}h \in \mathcal{G}^*\}$ *is compact and connected at a regular element* $h \in \mathcal{G}^*$;

ii) all functions $H_j{:}M^{2n} \to \mathbb{R}$, $j=\overline{1,k}$, *are functionally independent on* M^{2n};

iii)the Lie algebra $\mathcal{G}$ *of invariants satisfies the following relationship:*

$$\dim \mathcal{G} + rank\mathcal{G} = \dim M^{2n}, \tag{1.10}$$

where $rank\mathcal{G} = dim\ \mathcal{G}_h$ is *the dimension of a Cartan subalgebra* $\mathcal{G}_h \subset \mathcal{G}$. *Then the submanifold* $M_h^r \subset M^{2n}$ *is* $r = rank\mathcal{G}$ *-dimensional, invariant with respect each vector field* $K{\in}\ \Gamma(M^{2n})$, *generated by an element* $H \in \mathcal{G}_h$, *and diffeomorphic to the r-dimensional torus* $\mathbb{T}^r \simeq M_h^r$, *on which the motion is a quasiperiodic function of the evolution parameter* $t \in \mathbb{R}$. The simplest proof of the Mishchenko -Fomenko Theorem 1.6 can be obtained from the well known [3,16] classical Lie-Cartan theorem.

Theorem 1.7 *(S. Lie-E. Cartan) Spose that a point* $h{\in}\ \mathcal{G}^*$ *for a given Lie algebra* $\mathcal{G}$ *of invariants* $H_j : M^{2n} \to \mathbf{R}$, $j=\overline{1,k}$, *is not critical, and the rank* $||\{H_i, H_j\} : i,j = \overline{1,k}|| = 2(n-r)$ *is constant in an open neighborhood* $U_h \in \mathbb{R}^n$ *of the point* $\{h(H_j) = h_j \in \mathbb{R}{:}\ j = \overline{1,k}\} \subset \mathbb{R}^k$. *Then in the neighborhood* $(h \circ H)^{-1}: U_h \subset M^{2n}$ *there exist* $k{\in}\ \mathbb{Z}_+$ *independent functions* $f_s : \mathcal{G} \to \mathbb{R}$, $s = \overline{1,k}$, *such that the functions* $F_s := (f_s \circ H) : M^{2n} \in \mathbb{R}$, $s = \overline{1,k}$, *satisfy the following relationships:*

$$\{F_1, F_2\} = \{F_3, F_4\} = ... = \{F_{2(n-r)-1}, F_{2(n-r)}\} = 1, \tag{1.11}$$

with all other brackets $\{F_i, F_j\} = 0$, *where* $(i,j) \neq (2s-1, 2s)$, $s = \overline{1, n-r}$. *In particular,* $(k + r - n) \in \mathbb{Z}_+$ *functions* $F_j : M^{2n} \to \mathbb{R}$, $j = \overline{1, n-r}$, and $F_s : M^{2n} \to \mathbb{R}$, $s = \overline{1, k-2(n-r)}$, *compose an abelian algebra* $\mathcal{G}_\tau$ *of new invariants on* M^{2n}, *independent on* $(h \circ H)^{-1}(U_h) \subset M^{2n}$.

As a simple corollary of the Lie-Cartan Theorem 1.7 one obtains the following : in the case of the Mishchenko-Fomenko theorem when $rank\mathcal{G} + dim\mathcal{G} = dimM^{2n}$, that is $r + k = 2n$, the abelian algebra $\mathcal{G}_\tau$ (it is not a subalgebra of $\mathcal{G}$!) of invariants on M^{2n} is just $n = 1/2dimM^{2n}$-dimensional, giving rise to its local complete integrability in $(h \circ H)^{-1}(U_h) \subset$ M^{2n} via the abelian Liouville-Arnold theorem 1.5. It is also evident that the Mishchenko-Fomenko nonabelian integrability theorem 1.6 reduces to the commutative (abelian) Liouville-Arnold case when a Lie algebra

$\mathcal{G}$ of invariants is just abelian, since then $rank\mathcal{G} = dim\mathcal{G} = 1/2 \dim M^{2n} = n \in \mathbf{Z}_+$ - the standard complete integrability condition.

All the cases of integrability by quadratures described above pose the following fundamental question: How can one effectively construct by means of algebraic-analytical methods the corresponding integral submanifold imbedding

$$\pi_h : M_h^r \to M^{2n}, \tag{1.12}$$

where $r = dim\ rank\mathcal{G}$, thereby making it possible to express the solutions of an integrable flow on M_h^r as some exact quasi-periodic functions on the torus $\mathbb{T}^r \simeq M_h^r$.

Below we show examples of dynamical systems on a symplectic manifold $M^{\ 4}$ is diffeomorphic to the canonically symplectic cotangent phase space $T^*(\mathbb{R}^2) \simeq M^4$.

1.3. **Examples.** Here we consider some examples of investigations of integral submanifold imbedding mappings for abelian Liouville-Arnold integrable Hamiltonian systems on $T^*(\mathbb{R}^2)$.

1.3.1. *The Henon-Heiles system-1.* This flow is governed by the Hamiltonian

$$H_1 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + q_1 q_2^2 + \frac{1}{3}q_1^3 \tag{1.13}$$

on the canonically symplectic phase space $M^4 = T^*(\mathbb{R}^2)$ with the symplectic structure

$$\omega^{(2)} = \sum_{j=1}^{2} dp_j \wedge dq_j. \tag{1.14}$$

As is well known, there exists the following additional invariant that commutes with (2.1):

$$H_2 = p_1 p_2 + 1/3 q_2^3 + q_1^2 q_2, \tag{1.15}$$

that is $\{H_1, H_2\} = 0$ on the entire space M^4.

Take a regular element $h \in \mathcal{G} := \{H_j :\ M^4 \to \mathbb{R}: \quad j = \overline{1,2}\}$, with fixed values $h(H_j) = h_j \in \mathbb{R},\ j = \overline{1,2}$. Then the integral submanifold

$$M_h^2 := \{(q,p) \in M^4 : h(H_j) = h_j \in \mathbb{R},\ j = \overline{1,2}\}, \tag{1.16}$$

if compact and connected, is diffeomorphic to the standard torus $\mathbb{T}^2 \simeq \mathbb{S}^1 \times \mathbb{S}^1$ owing to the Liouville-Arnold theorem, and one can find cyclic (separable) coordinates $\mu_j \in \mathbb{S}^1,\ j = \overline{1,2}$, on the torus such that the symplectic structure (2.2) will take the form:

$$\omega^{(2)} = \sum_{j=1}^{2} dw_j \wedge d\mu_j\ , \tag{1.17}$$

where the conjugate variables $w_j \in T^*(\mathbb{S}^1)$, $j = \overline{1,2}$, on M_h^2 depend only on the corresponding variables $\mu_j \in \mathbb{S}_j^1,\ j = \overline{1,2}$. In this case it is evident that the evolution along M_h^2 will be separable and representable by means of quasi-periodic functions of the evolution parameters.

To show this, recall that the fundamental determining equations (2.34) based on the 1-forms $\bar{h}_j^{(1)} \in \Lambda(M_h^2)$, $j=\overline{1,2}$, satisfy the identity

$$\sum_{j=1}^{2} dH_j \wedge_j \bar{h}_j^{(1)} = \sum_{j=1}^{2} dp_j \wedge dq_j \ . \tag{1.18}$$

Here

$$\bar{h}_j^{(1)} = \sum_{k=1}^{2} \bar{h}_{jk}(q,p) dq_k, \tag{1.19}$$

where $j=\overline{1,2}$. Substituting (1.3.1) into (1.18), one obtains

$$\bar{h}_1^{(1)} = \frac{p_1 dq_1}{p_1^2 - p_2^2} + \frac{p_2 dq_2}{p_1^2 - p_2^2}, \ \ \bar{h}_2^{(1)} = \frac{p_2 dq_1}{p_{21}^2 - p_1^2} + \frac{p_1 dq_2}{p_1^2 - p_2^2} \ . \tag{1.20}$$

On the other hand, the following implication holds on $M_h^2 \subset M^4$:

$$\alpha_h^{(1)} = \sum_{j=1}^{2} w_j(\mu_j; h) d\mu_j \Rightarrow \sum_{j=1}^{2} p_j dq_j \ := \alpha^{(1)}, \tag{1.21}$$

where we have assumed that the integral submanifold M_h^2 admits the local coordinates in the base manifold $\mathbb{R}^2$ endowed with the canonical 1-form $\alpha_h^{(1)} \in \Lambda(M_h^2)$ as given in (1.20). Thus, making use of the imbedding $\pi_h : M_h^2 \to T^*(\mathbb{R}^2)$ in the form

$$q_j = q_j(\mu; h) \ , \quad p_j = p_j(\mu; h) \ , \tag{1.22}$$

$j=\overline{1,2}$, one readily finds that the equalities

$$p_j = \sum_{k=1}^{2} w_k(\mu_k; h) \ \ \partial\mu_k/\partial q_j \tag{1.23}$$

hold for $j=\overline{1,2}$ on the entire integral submanifold M_h^2.

Substituting (1.22) into (1.19) and using the corresponding characteristic relationships, one obtains after simple but cumbersome calculations the following differential-algebraic expressions:

$$\partial q_1/\partial\mu_1 - \partial q_2/\partial\mu_1 = 0, \ \ \partial q_1/\partial\mu_2 + \partial q_2/\partial\mu_2 = 0, \tag{1.24}$$

whose simplest solutions are

$$q_1 = (\mu_1 + \mu_2)/2 \ , \quad q_2 = (\mu_1 - \mu_2)/2 \ . \tag{1.25}$$

Using expressions (1.22) one finds that

$$p_1 = w_1 + w_2 \ , \quad p_2 = w_1 - w_2 \quad , \tag{1.26}$$

where the corresponding two algebraic curves $\Gamma_h^{(j)}$, $j=\overline{1,2}$ are given as

$$w_1 = \sqrt{h_1 + h_2 - 4/3\mu_1^3}, \qquad w_2 = \sqrt{h_1 - h_2 - 4/3\mu_2^3}. \tag{1.27}$$

Consequently, one obtains the separable [15] Hamiltonian functions (1.13) and (1.15) in a vicinity $U(M_h^2)$ of the integral submanifold M_h^2 :

$$h_1 = \frac{1}{2}w_1^2 + \frac{1}{2}w_2^2 + \frac{2}{3}(\mu_1^3 + \mu_2^3), \quad h_2 = \frac{1}{2}w_1^2 - \frac{1}{2}w_2^2 + \frac{2}{3}(\mu_1^3 - \mu_2^3), \tag{1.28}$$

which generate the following separable motions on $M_h^2 \subset T^*(\mathbb{R}^2)$:

$$d\mu_1/dt := \partial h_1/\partial w_1 = \sqrt{h_1 + h_2 - 4/3\mu_1^3}, \tag{1.29}$$

$$d\mu_2/dt := \partial h_1/\partial w_2 = \sqrt{h_1 - h_2 - 4/3\mu_2^3} \tag{1.30}$$

for the Hamiltonian (1.13), and

$$d\mu_1/dx := \partial h_2/\partial w_1 = \sqrt{h_1 + h_2 - 4/3\mu_1^3}, \tag{1.31}$$

$$d\mu_2/dt := \partial h_1/\partial w_2 = -\sqrt{h_1 - h_2 - 4/3\mu_2^3} \tag{1.32}$$

for the Hamiltonian (1.15), where $x, t \in \mathbb{R}$ are the corresponding evolution parameters.

1.3.2. *The Henon-Heiles system-2.* Analogously, one can show that there exists [28, 29] a similar to (1.24) and (1.25) integral submanifold imbedding for the following integrable modified Henon-Heiles involutive system:

$$H_1 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + q_1q_2^2 + \frac{16}{3}q_1^3, \tag{1.33}$$

$$H_2 = 9p_2^4 + 36q_1p_2^2q_2^2 - 12p_1p_2q_2^3 - 2q_2^4(q_2^2 + 6q_1^2) \quad , \tag{1.34}$$

where $\{H_1, H_1\} = 0$ on the entire phase space $M^4 = T^*(\mathbb{R}^2)$.

Based on considerations similar to the above, one can deduce the following [29] expressions:

$$q_1 = -\frac{1}{4}(\mu_1 + \mu_2) - \frac{3}{8}(\frac{w_1 + w_2}{\mu_1 - \mu_2})^2, \tag{1.35}$$

$$q_2^2 = -2\sqrt{h_2}/(\mu_1 - \mu_2), \quad w_1 = \sqrt{2/3\mu_1^3 - 4/3\sqrt{h_2} - 8h_1} \ , \tag{1.36}$$

$$p_1 = \frac{1}{2\sqrt{-6(\mu_1 + \mu_2 + 4q_1)}}[\frac{-2\sqrt{h_2}}{\mu_1 - \mu_2} - \mu_1\mu_2 + 4(\mu_1 + \mu_2)q_1 + 32q_1^2], \tag{1.37}$$

$$p_2 = \sqrt{h_2}(\mu_1 + \mu_2 + 4q_1)/(3(\mu_1 - \mu_2)) \ , \quad w_2 = \sqrt{2/3\mu_2^3 + 4/3\sqrt{h_2} - 8h_1} \ , \tag{1.38}$$

thereby solving explicitly the problem of finding the corresponding integral submanifold imbedding $\pi_h : M_h^2 \rightarrow T^*(\mathbb{R}^2)$ that generates separable flows in the variables $(\mu, w) \in T^*(M_h^2)$.

1.3.3. *A truncated four-dimensional Focker-Plank Hamiltonian system on $T^*(\mathbb{R}^2)$ and its integrability by quadratures.* Consider the following dynamical system on the canonically symplectic phase space $T^*(\mathbb{R}^2)$:

$$\left.\begin{array}{l} dq_1/dt = p_1 + \alpha(q_1+p_2)(q_2+p_1), \quad dq_2/dt = p_2, \\ dp_1/dt = -(q_1+p_2) - \alpha[q_2p_1 + 1/2(p_1^2+p_2^2+q_2^2)], \\ dp_2/dt = -(q_2+p_1), \end{array}\right\} = K_1(q,p), \tag{1.39}$$

where $K_1 : T^*(\mathbb{R}^2) \to T(T^*(\mathbb{R}^2))$ is the corresponding vector field on $T^*(\mathbb{R}^2) \ni (q,p)$, $t \in \mathbb{R}$ is an evolution parameter, called a truncated four-dimensional Focker-Plank flow. It is easy to verify that functions $H_j : T^*(\mathbb{R}^2) \to \mathbb{R}$, $j = \overline{1,2}$, where

$$H_1 = 1/2(p_1^2+p_2^2+q_1^2) + q_1p_2 + \alpha(q_1+p_2)[q_2p_1 + 1/2(p_1^2+p_2^2+q_2^2)] \tag{1.40}$$

and

$$H_2 = 1/2(p_1^2+p_2^2+q_2^2) + q_2p_1 \tag{1.41}$$

are functionally independent invariants with respect to the flow (1.39). Moreover, the invariant (1.40) is the Hamiltonian function for (1.39), that is the relationship

$$i_{K_1}\omega^{(2)} = -dH_1 \tag{1.42}$$

holds on $T^*(\mathbb{R}^2)$, where the symplectic structure $\omega^{(2)} \in \Lambda^2(T^*(\mathbb{R}^2))$ is given as follows:

$$\omega^{(2)} := d(pr^*\alpha^{(1)}) = \sum_{j=1}^{2} dp_j \wedge dq_j, \tag{1.43}$$

with $\alpha^{(1)} \in \Lambda^1(\mathbb{R}^2)$ to be the canonical Liouville form on $\mathbb{R}^2$:

$$\alpha^{(1)}(q;p) = \sum_{j=1}^{2} p_j dq_j \tag{1.44}$$

for any $(q,p) \in T^*(\mathbb{R}^2) \simeq \Lambda^1(\mathbb{R}^2)$.

The invariants (1.40) and (1.41) commute evidently with each other subject to the associated Poisson bracket on $T^*(\mathbb{R}^2)$:

$$\{H_1, H_2\} = 0. \tag{1.45}$$

Thereby, owing to the abelian Liouville-Arnold theorem [1, 3] the dynamical system (1.39) is completely integrable by quadratures on $T^*(\mathbb{R}^2)$, and we can apply the scheme devised before to the commuting invariants (1.40) and (1.41) subject to the symplectic structure (1.43). One easily calculates that

$$\omega^{(2)} = \sum_{i=1}^{2} dH_i \wedge h_i^{(1)}, \tag{1.46}$$

where the corresponding 1-forms $\pi_h^* h_i^{(1)} := \bar{h}_i^{(1)} \in \Lambda^1(M_h^2)$, $i = \overline{1,2}$, are given as

$$\begin{aligned} \bar{h}_1^{(1)} &= \frac{p_2dq_1 - (p_1+q_2)dq_2}{p_1p_2 - (p_1+q_2)(q_1+p_2) - \alpha h_2(p_1+q_2)}, \\ \bar{h}_2^{(1)} &= \frac{-[(q_1+p_2)(1+\alpha p_2)+\alpha h_2]dq_1 + (p_1+\alpha[h_2+(q_2+p_1)(\ q_1+p_2)])dq_2}{p_1p_2 - (q_2+p_1)(\alpha h_2 +\ q_1+p_2)}, \end{aligned} \tag{1.47}$$

and an invariant submanifold $M_h^2 \subset T^*(\mathbb{R}^2)$ is defined as

$$M_h^2 := \{(q,p) \in T^*(\mathbb{R}^2) \ : \ H_i(q,p) = h_i \in \mathbb{R}, i = \overline{1,2}\} \tag{1.48}$$

for some parameters $h \in \mathbb{R}^2$ and Based now on expressions (1.48), and (1.31) one can easily construct functions $\bar{P}_{ij}(w;h)$, $i,j=\overline{1,2}$, in (1.3.1), defined on $T^*(M_h^2) \simeq T^*(\otimes_{j=1}^2 \mathbb{S}_j^1)$ subject to the integral submanifold imbedding mapping $\pi_h : M_h^2 \to T^*(\mathbb{R}^2)$ in coordinates $\mu \in \otimes_{j=1}^2 \mathbb{S}_j^1 \subset \otimes_{j=1}^2 \Gamma_h^{(j)}$, which we don't write down in detail due to their a bit long and cumbersome form. Having applied then the criterion (1.20), we arrive at the following compatibility relationships subject to the mappings $q : (\otimes_{j=1}^2 \mathbb{S}_j^1) \times \mathbb{R}^2 \to \mathbb{R}^2$ and $p : (\otimes_{j=1}^2 \mathbb{S}_j^1) \times \mathbb{R}^2 \to T_q^*(\mathbb{R}^2)$:

$$
\begin{aligned}
&\partial q_1/\partial\mu_1 - \partial q_2/\partial\mu_2 = 0,\ w_1\partial\mathcal{L}_\mu/\partial w_1 - w_2\partial\mathcal{L}_\mu/\partial w_2 = 0,\\
&\partial^2 q_1/\partial\mu_2\partial h_2 + \partial^2 w_2/\partial\mu_2\partial h_2 = 0,\\
&\partial w_1/\partial h_1(\partial q_1/\partial h_1) = \partial w_2/\partial h_1(\partial q_2/\partial h_1),\\
&w_1\partial w_1/\partial h_1 - w_2\partial w_2/\partial h_2 = 0,\\
&\partial(w_1\partial w_1/\partial h_2)/\partial h_2 - \alpha^2\partial q_1/\partial\mu_1 = 0, ...
\end{aligned}
\tag{1.49}
$$

and so on,subject to variables $\mu \in \otimes_{j=1}^2 \mathbb{S}_j^1$ and $h \in \mathbb{R}^2$. Solving equations like (1.49) , one can find right away that the expressions

$$
\begin{aligned}
&p_1 = w_1, \quad p_2 = w_2,\\
&q_1 = c_1 + \mu_1 - w_2(\mu_2;h),\\
&q_2 = c_2 + \mu_2 - w_1(\mu_1;h),\\
&\mathcal{L}_\mu(h) = -w_1 w_2,
\end{aligned}
\tag{1.50}
$$

where $c_j(h_1,h_2) \in \mathbb{R}^1$, $j=\overline{1,2}$, are constant, hold on $T^*(M_h^2)$, giving rise to the following Picard- Fuchs type equations in the form (1.21):

$$
\begin{aligned}
&\partial w_1(\mu_1;h)/\partial h_1 = 1/w_1,\\
&\partial w_1(\mu_1;h)/\partial h_2 = \alpha^2 h_2/w_1,\\
&\partial w_2(\mu_2;h)/\partial h_1 = 0\\
&\partial w_2(\mu_2;h)/\partial h_2 = 1/w_2.
\end{aligned}
\tag{1.51}
$$

The Picard-Fuchs equations (1.51) can be easily integrated by quadratures as follows:

$$
\begin{aligned}
&w_1^2 + k_1(\mu_1) - \alpha^2 h_2 - 2h_1 = 0,\\
&w_2^2 + k_2(\mu_2) - 2h_2 = 0,
\end{aligned}
\tag{1.52}
$$

where $k_j : \mathbb{S}_j^1 \to \mathbb{C}$, $j=\overline{1,2}$, are still unknown functions. For them to be determined explicitly, it is necessarily to substitute (1.50) into expressions (1.40) and (1.41), making use of (1.52) that amounts to the following results:

$$
k_1 = \mu_1^2, \quad k_2 = \mu_2^2 \tag{1.53}
$$

under the condition that $c_1 = -\alpha h_2$, $c_2 = 0$. Thereby, we have constructed owing to (1.52) the corresponding algebraic curves $\Gamma_h^{(j)}$, $j=\overline{1,2}$, in the explicit form:

$$
\begin{aligned}
&\Gamma_h^{(1)} := \{(\lambda, w_1) : w_1^2 + \lambda^2 - \alpha^2 h_2^2 - 2h_1) = 0\},\\
&\Gamma_h^{(2)} := \{(\lambda, w_2) : w_2^2 + \lambda^2 - 2h_2 = 0\},
\end{aligned}
\tag{1.54}
$$

where $(\lambda, w_j) \in \mathbb{C} \times \mathbb{C}$, $j = \overline{1,2}$, and $h \in \mathbb{R}^2$ are arbitrary parameters. Making use now expressions (1.55) and (1.50), one can construct in explicit form the integral submanifold imbedding mapping $\pi_h : M_h^2 \to T^*(\mathbb{R}^2)$ for the flow (1.39):

$$
\begin{aligned}
q_1 &= \mu_1 - \sqrt{2h_2 - \mu_2^2} - \alpha h_2^2, \quad p_1 = w_1(\mu_1; h), \\
q_2 &= \mu_2 - \sqrt{2h_1 - \alpha^2 h_2^2 - \mu_1^2}, \quad p_2 = w_2(\mu_2; h),
\end{aligned} \tag{1.55}
$$

where $(\mu, w) \in \otimes_{j=1}^2 \Gamma_h^{(j)}$. As was mentioned before in chapter 2, formulas (1.55) together with explicit expressions (**??**) make it possible right away to find solutions to the truncated Focker-Plank flow (1.39) by quadratures, thereby completing its integrability.

==

2. The Poisson structures and Lie group actions on manifolds with principal bundle structure

2.1. **introduction.** It was understood during recent decades that many dynamical systems of classical physics and mechanics are endowed with the symplectic structures [1, 3, 2] and associated Poisson brackets. In many such cases the structure of the Poisson bracket appear to be canonical and is given on the dual space of the corresponding Lie algebra of symmetries, being augmented in some cases with a 2-cocycle, and sometimes having a gauge nature. These observations gave rise to a deep group-theoretical interpretation of these Poisson structures for many dynamical systems of mathematical physics, especially for the completely integrable ones.

The investigation of dynamical systems possessing a rich internal symmetry structure is usually carried out in three steps: 1) determining the symplectic structure (the Poisson bracket), recasting the initial dynamical system into Hamiltonian form; 2) determining conservation laws (invariants or constants of the motion) in involution; 3) determining an additional set of variables and computing their evolution under the action of Hamiltonian flows, associated with the invariants.

In many cases the above program is too difficult to realize because of the lack of regular methods for seeking both symplectic structures and a system of the related invariants. Thereby, of particular interest there present those dynamical systems with a deep intrinsic group nature for which there exists a possibility of investigating their symmetry structure in exact form. The corresponding symplectic manifolds on which these systems are, in general, pull-backs of the corresponding group actions, related with the coadjont action of a Lie group G on the dual space $\mathcal{G}^*$ to its Lie algebra $\mathcal{G}$ together with the natural Poisson structure upon them. In many cases these spaces carry a principle fiber bundle structure and can be endowed with some connections, playing a very important role for describing the symmetry structure of related dynamical systems.

2.2. **The Lie group actions on Poisson manifolds and the orbit structure.** Let us recall some definitions. The *Poisson structure* on a smooth manifold M is given by the pair $(M, \{.,.\})$, where

$$
\{.,.\} : \quad D(M) \times D(M) \to D(M), \tag{2.1}
$$

is a Poisson bracket mapping onto the space of real-valued smooth functions on M, satisfying the conditions: 1) it is bilinear and skew-symmetric; 2) it is a differentiation with respect to each of the argument; 3) it obeys the Jacobi identity. Any function $H \in D(M)$ determines the vector field sgrad H (*symplectic gradient*) for all $f \in D(M)$ via the formula:

$$\text{sgrad}\, H(f) := \{H, f\}. \tag{2.2}$$

The vector field $\text{sgrad}\, H: \ M \to T(M)$ is called Hamiltonian, with the Hamiltonian function $H \in D(M)$.

A symplectic structure $\omega^{(2)}$ on M supplies the manifold M with the Poisson bracket in the natural manner. For any function $H \in D(M)$ the vector field $\text{sgrad}\, H: \ M \to T(M)$ is defined via the rule:

$$i_{\text{sgrad}\, H}\omega^{(2)} := -dH, (1.3) \tag{2.3}$$

whence

$$\{H, f\} := -\omega^{(2)}(\ \text{sgrad}\, H,\ \text{sgrad}\, f\ ) \tag{2.4}$$

for all $f \in D(M)$, where $i_{\text{sgrad}\, f}\omega^{(2)} := -df$ by definition.

The Poisson or its associated Hamiltonian structure forms naturally a wider class than do symplectic ones. It is not hard to convince ourselves that any Poisson structure $\{.,.\}$ on a manifold M is stratified by symplectic structures.

The next class of Poisson structures appeared to be very important for applications [36, 5, 2, 45, 39, 44]. Let G be a connected real Lie group, $\mathcal{G}$ its Lie algebra over the field $\mathbf{R}$, and $\mathcal{G}^*$ its linear space dual to $\mathcal{G}$. To each element $x \in \mathcal{G}^*$ there is the associated endomorphism $ad\, x: \ \mathcal{G} \to \mathcal{G},\ ad\ x(y) := [x, y],\ y \in \mathcal{G}$, where $[.,.]$ – the Lie structure of the Lie algebra $\mathcal{G}$. To each element $X \in G$ there is the associated automotphism $Ad\ X: \ \mathcal{G} \to \mathcal{G}$ via the rule:

$$Ad\ X: \ y \to dl_{X^*} \circ dr_X(y), \tag{2.5}$$

where $y \in \mathcal{G}$, dl_X and dr_X are the tangent mappings for left and right translations on the Lie group G, respectively.

Denote by ad^* and Ad^* adjoint mappings to ad and Ad, respectively, on $\mathcal{G}$. Then for all $\alpha \in \mathcal{G}^*$, $x, y \in \mathcal{G}$ the following identity obtains:

$$\langle\ ad^*\ x(\alpha), y\ \rangle := \langle\ \alpha, [x, y]\ \rangle, \tag{2.6}$$

where $\langle .,. \rangle$ is the convolution of $\mathcal{G}^*$ with $\mathcal{G}$.

The representation ad^* of the Lie algebra $\mathcal{G}$ and Ad^* of the Lie group G in the space $\text{End}\, \mathcal{G}^*$ are called co-adjoint.

Let $f \in D(\mathcal{G}^*)$; then one can determine the gradient $\nabla f: \ \mathcal{G}^* \to \mathcal{G}^*$ via the rule:

$$\langle\ m, \nabla f(\alpha)\ \rangle := df(\alpha; m) := \frac{d}{d\varepsilon} f(\alpha + \varepsilon m)|_{\varepsilon=0}, \tag{2.7}$$

where $(\alpha; m) \in \mathcal{G}^* \times \mathcal{G} \cong T(G^*)$. The structure of the Poisson bracket on $\mathcal{G}^*$ is defined as follows:

$$\{f, g\}(\alpha) := \langle\ \alpha,\ [\nabla f(\alpha), \nabla g(\alpha)]\ \rangle \tag{2.8}$$

for any $f, g \in D(\mathcal{G}^*)$. A proof that the bracket (2.8) is a Poisson bracket, is given making use of standard [2, 5, 36] considerartions. The corresponding Hamiltonian vector field for a function $H \in D(\mathcal{G}^*)$ takes the form:

$$\text{sgrad}\, H(\alpha) = (\ \alpha; ad^*\ \nabla H(\alpha)(\alpha)\ ) \in T(G^*), \tag{2.9}$$

where $\alpha \in \mathcal{G}^*$ is arbitrary. The vector field $\text{sgrad} H(\alpha)$ is tangent to the orbit $O_\alpha(G)$ of the Lie group G through an element $\alpha \in \mathcal{G}$ under the Ad^*-action. These orbits are symplectic strata of the manifold M. To each element $u_\nu := (\ \alpha; ad^*\ x_\nu(\alpha)) \in T_\alpha(O_\alpha) \equiv V_\alpha$, $\nu = \overline{1, 2}$, define

$$\omega_\alpha(u_1, u_2) := \langle\ \alpha,\ [x_1, x_2]\ \rangle. \tag{2.10}$$

Then obviously, ω_α is a symplectic structure on O_α for all $\alpha \in \mathcal{G}^*$. Thereby one obtains that the above symplectic stratification of the Poisson structure (2.1) is realized by means of $Ad^*\ G$, the expansion in orbits in the space $\mathcal{G}^*$. Notice here that each orbit $O_\alpha(G)$, $\alpha \in \mathcal{G}^*$, is an uniform symplectic submanifold in $\mathcal{G}^*$; i.e., the action Ad^* of the Lie group is symplectic and transitive. The restriction of the vector field sgrad H upon an orbit O_α is defined uniquely via the restriction of the Hamiltonian function $H \in D(\mathcal{G}^*)$ upon the orbit $O_\alpha(G)$, $\alpha \in \mathcal{G}^*$.

2.3. **The canonical reduction method on canonically symplectic spaces and related geometric structures on principal fiber bundles: introductory backgrounds.** The canonical reduction method in application to many geometric objects on symplectic manifolds with symmetry appears to be very effective tool for their studying, in particular for finding the effective phase space variables [5, 36, 37, 38, 39, 43] on integral submanifolds of Hamiltonian dynamical systems in which they are integrable [5, 36, 2, 48] via the Liouville-Arnold theorem, for investigating related stability problems [5, 40, 46] of Hamiltonian dynamical systems under small perturbations and so on.

Let G denote a given Lie group with the unity element $e \in G$ and the corresponding Lie algebra $\mathcal{G} \simeq T_e(G)$. Consider a principal fiber bundle $M(N; G)$ with the projection $p : M \to N$, the structure group G and base manifold N, on which the Lie group G acts [5, 36, 2, 47] by means of a smooth mapping $\varphi : M \times G \to M$. Namely, for each $g \in G$ there is a group of diffeomorphisms $\varphi_g : M \to M$, generating for any fixed $u \in M$ the following induced mapping: $\hat{u} : G \to M$, where

$$\hat{u}(g) = \varphi_g(u). \tag{2.11}$$

On the principal fiber bundle $p : (M, \varphi) \to N$ there is assigned [37, 41, 47, 2] a connection $\Gamma(\mathcal{A})$ by means of such a morphism $\mathcal{A}$: : $(T(M), \varphi_{g,*}) \to (\mathcal{G}, Ad_{g^{-1}})$, that for each $u \in M$ the mapping $\mathcal{A}(u) : T_u(M) \to \mathcal{G}$ is a left inverse one to the mapping $\hat{u}_*(e) : \mathcal{G} \to T_u(M)$ and the mapping $\mathcal{A}^*(u) : \mathcal{G}^* \to T_u^*(M)$ is a right inverse one to the mapping $\hat{u}^*(e) : T_u^*(M) \to \mathcal{G}^*$, that is

$$\mathcal{A}(u)\hat{u}_*(e) = 1, \quad \hat{u}^*(e)\mathcal{A}^*(u) = 1. \tag{2.12}$$

As usually, denote by $\varphi_g^* : T^*(M) \to T^*(M)$ the corresponding lift of the mapping $\varphi_g : M \to M$ at any $g \in G$. If $\alpha^{(1)} \in \Lambda^1(M)$ is the canonical G - invariant 1-form on M, the canonical symplectic structure $\omega^{(2)} \in \Lambda^2(T^*(M))$ given by

$$\omega^{(2)} := d\, pr_M^* \alpha^{(1)} \tag{2.13}$$

generates the corresponding momentum mapping $l : T^*(M) \to \mathcal{G}^*$, where

$$l(\alpha^{(1)})(u) = \hat{u}^*(e)\alpha^{(1)}(u) \tag{2.14}$$

for all $u \in M$. Remark here that the principal fiber bundle structure $p : M \to N$ means in part the exactness of the following two adjoint sequences of mappings:

$$0 \leftarrow \mathcal{G} \overset{\hat{u}^*(e)}{\leftarrow} T_u^*(M) \overset{p^*(u)}{\leftarrow} T_{p(u)}^*(N) \leftarrow 0, \tag{2.15}$$

$$0 \to \mathcal{G} \overset{\hat{u}_*(e)}{\to} T_u(M) \overset{p_*(u)}{\to} T_{p(u)}(N) \to 0, \tag{2.16}$$

that is

$$p_*(u)\hat{u}_*(e) = 0, \quad \hat{u}^*(e)p^*(u) = 0 \tag{2.17}$$

for all $u \in M$. Combining (2.17) with (2.12) and (2.14), one obtains such an embedding:

$$[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u) \in range\ p^*(u) \tag{2.18}$$

for the canonical 1-form $\alpha^{(1)} \in \Lambda^1(M)$ at $u \in M$. The expression (2.18) means, of course, that

$$\hat{u}^*(e)[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u) = 0 \tag{2.19}$$

for all $u \in M$. Taking now into account that the mapping $p^*(u) : T_{p(u)}^*(N) \to T_u^*(M)$ is for each $u \in M$ injective, it has the unique inverse mapping $(p^*(u))^{-1}$ upon its image $p^*(u)T_{p(u)}^*(N) \subset T_u^*(M)$. Thereby for each $u \in M$ one can define a morphism $p_{\mathcal{A}} : (T^*(M), \varphi_g^*) \to (T^*(N), id)$ as

$$p_{\mathcal{A}}(u) : \alpha^{(1)}(u) \to (p^*(u))^{-1}[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\alpha^{(1)}(u). \tag{2.20}$$

Based on the definition (2.20) one can easily check that the diagram

$$\begin{array}{ccc} T^*(M) & \overset{p_{\mathcal{A}}}{\to} & T^*(N) \\ pr_M\downarrow & & \downarrow pr_N \\ M & \overset{p}{\to} & N \end{array} \tag{2.21}$$

is commutative.

Let now an element $\xi \in \mathcal{G}^*$ be G-invariant, that is $Ad_{g^{-1}}^*\xi = \xi$ for all $g \in G$. Denote also by $p_{\mathcal{A}}^{\xi}$ the restriction of the mapping (2.20) upon the subset $\mathcal{M}_\xi := l^{-1}(\xi) \in T^*(M)$, that is the mapping $p_{\mathcal{A}}^{\xi} : \mathcal{M}_\xi \to T^*(N)$, where for all $u \in M$

$$p_{\mathcal{A}}^{\xi}(u) : \mathcal{M}_\xi \to (p^*(u))^{-1}[1 - \mathcal{A}^*(u)\hat{u}^*(e)]\mathcal{M}_\xi. \tag{2.22}$$

Now one can characterize the structure of the reduced phase space $\bar{\mathcal{M}}_\xi := \mathcal{M}_\xi/G$ by means of the following simple lemma.

Lemma 2.1. *The mapping* $p^{\xi}_{\mathcal{A}}(u) : \mathcal{M}_{\xi} \to T^{*}(N)$ *is a principal fiber G-bundle with the reduced space* $\bar{\mathcal{M}}_{\xi} = \mathcal{M}_{\xi})/G$ *being diffeomorphic to* $T^{*}(N)$.

Denote by $< .,. >_{\mathcal{G}}$ the standard Ad-invariant nondegenerate scalar product on $\mathcal{G}^{*} \times \mathcal{G}$. Based on Lemma 1.1 one derives the following characteristic theorem.

Theorem 2.2. *Given a principal fiber G bundle with a connection* $\Gamma(\mathcal{A})$ *and an G invariant element* $\xi \in \mathcal{G}^{*}$, *then every such connection* $\Gamma(\mathcal{A})$ *defines a symplectomorphism* $\nu_{\xi} : \bar{\mathcal{M}}_{\xi} \to T^{*}(N)$ *between the reduced phase space* $\bar{\mathcal{M}}_{\xi}$ *and cotangent bundle* $T^{*}(N)$. *Moreover, the following equality*

$$(p^{\xi}_{\mathcal{A}})(d\ pr^{*}_{N}\beta^{(1)} + pr^{*}_{N}\ \Omega^{(2)}_{\xi}) = d\ pr^{*}_{M_{\xi}}\alpha^{(1)} \tag{2.23}$$

holds for the canonical 1-form $\beta^{(1)} \in \Lambda^{(1)}(N)$ *and* $\alpha^{(1)} \in \Lambda^{(1)}(M)$, *where* $M_{\xi} := pr_{M}\mathcal{M}_{\xi} \subset M$, *2-form* $\Omega^{(2)}_{\xi} \in \Lambda^{(2)}(N)$ *is the* ξ *-component of the corresponding curvature 2-form* $\Omega^{(2)} \in \Lambda^{(2)}(M) \otimes \mathcal{G}$.

Proof. One has that on $\mathcal{M}_{\xi} \subset T^{*}(M\ )$ the following expression due to (2.20) holds:

$$p^{*}(u)p^{\xi}_{\mathcal{A}}(\alpha^{(1)}(u)) := p^{*}(u)\beta^{(1)}(pr_{N}(u)) = \alpha^{(1)}(u) - \mathcal{A}^{*}(u)\hat{u}^{*}(e)\alpha^{(1)}(u) \tag{2.24}$$

for any $\beta^{(1)} \in T^{*}(N), \alpha^{(1)} \in \mathcal{M}_{\xi}$ and $u \in M_{\xi}$. Thus we get easily that for such $\alpha^{(1)} \in \mathcal{M}_{\xi}$ there holds

$$\alpha^{(1)}(u) = (p^{\xi}_{\mathcal{A}})^{-1}\beta^{(1)}(p_{N}(u)) = p^{*}(u)\beta^{(1)})(pr_{N}(u)) + < \mathcal{A}(u), \xi >_{\mathcal{G}} \tag{2.25}$$

for all $u \in M_{\xi}$. Recall now that in virtue of (2.21) one gets on M_{ξ} and $\mathcal{M}_{\xi}$ the following relationships:

$$p \cdot\ pr_{M_{\xi}} = pr_{N} \cdot\ p^{\xi}_{\mathcal{A}},\ \ pr^{*}_{M_{\xi}} \cdot\ p^{*}\ =\ (p^{\xi}_{\mathcal{A}})^{*} \cdot\ pr^{*}_{N}\ . \tag{2.26}$$

Therefore we can write down now that for any $u \in M$

$$\begin{aligned} pr^{*}_{M_{\xi}}\alpha^{(1)}(u) &= pr^{*}_{M_{\xi}}p^{*}(u)\beta^{(1)}(p(u)) + pr^{*}_{M_{\xi}} < \mathcal{A}(u), \xi > \\ &= (p^{\xi}_{\mathcal{A}})^{*}pr^{*}_{N}\beta^{(1)}(u) + pr^{*}_{M_{\xi}} < \mathcal{A}(u), \xi >, \end{aligned}$$

whence taking the external differential, one arrives at the following equality:

$$\begin{aligned} d\ pr^{*}_{M_{\xi}}\alpha^{(1)}(u) &= (p^{\xi}_{\mathcal{A}})^{*}d(pr^{*}_{N}\beta^{(1)})(u) + pr^{*}_{M_{\xi}} < d\ \mathcal{A}(u), \xi > \\ &= (p^{\xi}_{\mathcal{A}})^{*}d(pr^{*}_{N}\beta^{(1)})(u) + pr^{*}_{M_{\xi}} < \Omega^{(2)}(u)), \xi > \\ &= (p^{\xi}_{\mathcal{A}})^{*}d(pr^{*}_{N}\beta^{(1)})(u) + pr^{*}_{M_{\xi}}p^{*} < \Omega^{(2)}, \xi > (u) \\ &= (p^{\xi}_{\mathcal{A}})^{*}d(pr^{*}_{N}\beta^{(1)})(u) + (p^{\xi}_{\mathcal{A}})^{*}pr^{*}_{N} < \Omega^{(2)}, \xi > (u) \\ &= (p^{\xi}_{\mathcal{A}})^{*}[d(pr^{*}_{N}\beta^{(1)})(u) + pr^{*}_{N}\Omega^{(2)}_{\xi}(u)]. \end{aligned}$$

When deriving the above expression we made use of the following property satisfied by the curvature 2-form $\Omega^{(2)} := d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \in \Lambda^{2}(M) \otimes \mathcal{G}$:

$$\begin{aligned} &< d\mathcal{A}(u), \xi >_{\mathcal{G}} = < d\mathcal{A}(u) + \mathcal{A}(u) \wedge \mathcal{A}(u), \xi >_{\mathcal{G}} \\ - &< \mathcal{A}(u) \wedge \mathcal{A}(u), \xi >_{\mathcal{G}} = < \Omega^{(2)}(u), \xi >_{\mathcal{G}} = \\ = &< \Omega^{(2)}(u), Ad^{*}_{g}\xi >_{\mathcal{G}} = < Ad_{g}\Omega^{(2)}(u), \xi >_{\mathcal{G}} = \\ = &< \Omega^{(2)}, \xi > (p(u))_{\mathcal{G}} := p^{*}\Omega^{(2)}_{\xi}(u) \end{aligned}$$

at any $u \in M$, since for any $A, B \in \mathcal{G}$ there holds $< [A,B], \xi >_{\mathcal{G}} = < B, ad^*_A \xi >_{\mathcal{G}} = 0$ in virtue of of the invariance condition $Ad^*_g \xi = \xi$ for any $g \in G$. Thereby the proof is finished.▷

Remark 2.3. *As the canonical 2-form* $d\ pr^*_M \alpha^{(1)} \in \Lambda^{(2)}(T^*(M))$ *is G-invariant on* $T^*(M)$ *due to the construction, it is evident that its restriction upon the G-invariant submanifold* $\mathcal{M}_\xi \subset T^*(M)$ *will be effectively defined only on the reduced space* $\bar{\mathcal{M}}_\xi$*, that ensures the validity of the equality sign in (2.23).*

As a simple but useful consequence of Theorem 1.2 one can formulate the following useful enough for applications results.

Theorem 2.4. *Let a momentum mapping value* $l(\alpha^{(1)})(u) = \hat{u}^*(e)\alpha^{(1)}(u) = \xi \in \mathcal{G}^*$ *has the isotropy group* G_ξ *acting naturally on the subset* $\mathcal{M}_\xi \subset T^*(M)$ *invariantly, freely and properly, so that the reduced phase space* $(\bar{\mathcal{M}}_\xi\ , \bar{\omega}^{(2)}_\xi)$ *is symplectic, where by definition* [1, 6]*, for the natural embedding mapping* $\pi_\xi : \mathcal{M}_\xi \to T^*(M)$ *and the reduction mapping* $r_\xi : \mathcal{M}_\xi \to \bar{\mathcal{M}}_\xi$ *the defining equality*

$$r^*_\xi \bar{\omega}^{(2)}_\xi := \pi^*_\xi (d\ pr^*_M \alpha^{(1)}) \tag{2.27}$$

holds on $\mathcal{M}_\xi$*. If an associated principal fiber bundle* $p : M \to N$ *has a structure group coinciding with* G_ξ*, then the reduced symplectic space* $(\bar{\mathcal{M}}_\xi, \bar{\omega}^{(2)}_\xi)$ *is symplectomorphic to the cotangent symplectic space* $(T^*(N), \sigma^{(2)}_\xi)$*, where*

$$\sigma^{(2)}_\xi = d\ pr^*_N \beta^{(1)} + pr^*_N \Omega^{(2)}_\xi, \tag{2.28}$$

and the corresponding symplectomorphism is given by the relation like (2.23) .

Concerning some applications the following criterion can be useful when constructing associated fibre bundles with connections related with the symplectic structure reduced on the space $\bar{\mathcal{M}}_\xi$.

Theorem 2.5. *In order that two symplectic spaces* $(\bar{\mathcal{M}}_\xi, \bar{\omega}^{(2)}_\xi)$ *and* $(T^*(N), d\ pr^*_N \beta^{(1)})$ *were symplectomorphic, it is necessary and sufficient that the element* $\xi \in \ker\ h$*, where for G-invariant element* $\xi \in \mathcal{G}^*$ *the mapping* $h : \xi \to [\Omega^{(2)}_\xi] \in H^2(N;\mathbb{Z})$ *with* $H^2(N;\mathbb{Z})$ *being the cohomology group of 2-forms on the manifold N.*

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3. A new integrable Whitham type nonlinear evolution equation describing short-wave perturbations in a relaxing medium: Lagrangian and Hamiltonian analysis

3.1. **Introduction.** A shortwave perturbations in a relaxing medium can be modeled by means of a Whitham type evolution equation

$$du/dt = 2uu_x + \int_{\mathbb{R}} \mathcal{K}(x,s) u_s ds, \tag{3.1}$$

discussed first in [50]. Here the kernel $\mathcal{K} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ depends on the medium elasticity properties with spacial memory and can, in general, be a function of the pressure gradient $u_x \in C^2(\mathbb{R};\mathbb{R})$, evolving in respect to the equation (3.1). In particular, if $\mathcal{K}(x,s) = \frac{1}{2} \mid x - s \mid,\ x,s \in \mathbb{R}$, then the equation (3.1) reduces to

$$du/dt = 2uu_x + \partial^{-1}u, \tag{3.2}$$

which was studied before in [51, 52, 49].

Since some media possess elasticity properties depending strongly on the spatial pressure gradient $u_x,\ x \in \mathbb{R}$, the corresponding Whitham kernel looks like

$$\mathcal{K}(x,s) := -\theta(x-s)u_s \tag{3.3}$$

for $x,s \in \mathbb{R}$, naturally modelling the relaxing spacial memory effects. The resulting equation (3.1) with the kernel () becomes as

$$du/dt = 2uu_x - \partial^{-1}u_x^2 := K[u], \tag{3.4}$$

which appears to possess very interesting mathematical properties. The latter will be the main topic of the next sections following below.

3.2. **Lagrangian analysis.** A more mathematically correct form of the equation (3.4) looks like

$$u_{xt} = 2(uu_x)_x - u_x^2, \tag{3.5}$$

being a nonlinear hyperbolic flow on the axis $\mathbb{R}$. Concerning the preceding form (3.4) it is necessary to define the operation $\partial^{-1} : C(\mathbb{R};\mathbb{R}) \to C(\mathbb{R};\mathbb{R})$. If one take into consideration the class $C_{2\pi}^{\infty}(\mathbb{R};\mathbb{R}) \subset C(\mathbb{R};\mathbb{R})$ of 2π- periodic solutions to the equation (3.5), then one can accept that

$$\partial^{-1}(\cdot) := \frac{1}{2}[\int_0^x (\cdot)ds - \int_x^{2\pi} (\cdot)ds], \tag{3.6}$$

satisfying the defining property $\partial \cdot \partial^{-1} = 1$ for all $x \in [0,2\pi]$. Thereby, for convenience, we will consider the flow (5.4) as that in the periodic smooth functional manifold $M := C_{2\pi}^{\infty}(\mathbb{R};\mathbb{R})$. The corresponding vector field $K : M \to T(M)$ defines on M a dynamical system, which appears to possess both Lagrangian and Hamiltonian properties.

To demonstrate them in detail, consider the partial differential equation (3.5) and prove that it is of Lagrangian form, that is

$$u_{xt} = -\frac{\delta H_\vartheta}{\delta u} := \xi[u], \tag{3.7}$$

where $H_\vartheta : M \to \mathbb{R}$ is some Fréchet smooth Lagrangian function. For proving (3.7), following the scheme in [53], it is enough to state only that there holds the Volterrian identity $\xi^{'} = \xi^{'*}$, that is

$$[2(uu_x)_x - u_x^2]' = [2(uu_x)_x - u_x^2]'^*, \tag{3.8}$$

where the sign "$\prime$" means the Fréchet derivative with respect to the variable $u \in M$ and " $*$ " means the corresponding conjugation with respect to the natural scalar product on the tangent space $T(M) \simeq T^*(M)$. As a result, there exists a Lagrangian function $H_\vartheta : M \to \mathbb{R}$ in the following explicit form:

$$H_\vartheta := \int_0^{2\pi} \mathcal{H}_\vartheta dx = \int_0^{2\pi} uu_x^2 dx. \tag{3.9}$$

Thus, the expression (3.7) can be presented as the Euler equation

$$\delta\mathcal{L}/\delta u = 0, \tag{3.10}$$

where, by definition,

$$\mathcal{L} := \int_0^t \int_0^{2\pi} (\frac{1}{2}u_x u_\tau - \mathcal{H}_\vartheta)dxd\tau. \tag{3.11}$$

Recall now, that owing to the results in [53], any Lagrangian system in the form (3.10) is Hamiltonian. To show this, rewrite the action functional (3.11) as

$$\mathcal{L} = \int_0^t [(\varphi, u_\tau) - H_\vartheta]d\tau, \tag{3.12}$$

where $\varphi := (1/2)u_x \in T^*(M)$. Then the condition (3.10) gives rise to the equality

$$u_t = -\vartheta \; grad \; H_\vartheta = K[u], \tag{3.13}$$

where, by definition,

$$\vartheta^{-1} := \varphi\prime - \varphi'^* = \partial/\partial x. \tag{3.14}$$

As it is easy to see, the operator $\vartheta := \partial^{-1} : T^*(M) \to T(M)$ is necessary implectic [53, 65] and with respect to the flow (3.2) also Noetherian. Thus we have stated the following theorem.

Theorem 3.1. *The partial differential equation (3.5) is equivalent on the functional manifold M to the Hamiltonian flow (3.2) with the Hamiltonian function (3.9) and co-implectic structure (3.14).*

This result means that our flow (3.4) on M, being Hamiltonian, is conservative, thereby one can expect it possesses also additional hidden conservation laws, which can be important for its integrability analysis. This assumption, as we shall show below, appears to hold really.

3.3. **Gradient-holonomic analysis.** Since any conservation law $\gamma \in D(M)$ satisfies the linear Lax equation

$$d\psi/dt + K'^{+}\psi = 0, \tag{3.15}$$

where $\psi = grad \; \gamma \in T^*(M)$, under the condition of its existence in the form of a local functional on M, it can be found for instance, by means of the asymptotic small parameter method [53]. In particular, one easily gets that expressions

$$\psi_\vartheta = u_{xx}, \quad \psi_{\eta_{-1}} = \frac{1}{2}(u_x^2 - (u)_{xx}^2) \tag{3.16}$$

satisfy the Lax equation (3.15) and are the gradients of the corresponding functionals on M, that is

$$\psi_\vartheta = grad \; \gamma_\vartheta \quad \psi_{\eta_{-1}} = grad \; \gamma_{\eta_{-1}}, \tag{3.17}$$

where

$$\gamma_\vartheta = \frac{1}{2}\int_0^{2\pi} u_x^2 dx \quad \gamma_{\eta_{-1}} = \frac{1}{2}\int_0^{2\pi} u u_x^2 dx. \tag{3.18}$$

Thus, we have stated that our dynamical system (3.4) allows additional invariants (conservation laws), which can be used within the gradient-holonomic algorithm [53] for finding new associated nontrivial implectic structures on the manifold M. Namely, let us represent conservation laws (3.16) in the scalar product form on M as

$$\gamma_\vartheta = (\varphi_\vartheta, u_x) \quad \gamma_{\eta_{-1}} = (\varphi_{\eta_{-1}}, u_x), \tag{3.19}$$

where

$$\varphi_\vartheta = \frac{1}{2}u_x, \quad \varphi_{\eta_{-1}} = -\frac{1}{2}\partial^{-1}u_x^2 \in T^*(M). \tag{3.20}$$

Then operators

$$\vartheta^{-1} = \varphi_\vartheta^{'} - \varphi_\eta^{'*} = \frac{1}{2}\partial - (-\frac{1}{2}\partial) = \partial, \tag{3.21}$$

$$\eta_{-1}^{-1} = \varphi_{\eta_{-1}}^{'} - \varphi_{\eta_{-1}}^{'*} = \partial^{-1}u_{xx} + u_{xx}\partial^{-1} \tag{3.22}$$

will be co-implectic [53, 65] on M, and, as it is easy to check, also Noetherian with respect to our dynamical system (3.4). Moreover, via direct calculations one can show that the corresponding implectic operators $\vartheta,\ \eta_{-1} : T^*(M) \to T(M)$ are compatible on M, that is for any $\lambda \in \mathbb{R}$ the expression $\vartheta + \lambda\eta_{-1}$ is implectic too on M [53, 54]. This means, in particular, that all operators of the form

$$\eta_n = \vartheta(\eta_{-1}^{-1}\vartheta)^n \tag{3.23}$$

for $n \in \mathbb{Z}$ will be implectic too on M. Another consequence from this fact is the existence of an infinite hierarchy of invariants $\gamma_n \in D(M),\ n \in \mathbb{Z}$, satisfying the expressions

$$K[u] = -\eta_n \ grad\ \gamma_n. \tag{3.24}$$

As a particular case one can define an implectic operator $\eta : T^*(M) \to T(M)$ in the form

$$\eta = \vartheta\eta_{-1}^{-1}\vartheta = \partial^{-2}u_{xx}\partial^{-1} + \partial^{-1}u_{xx}\partial^{-2}. \tag{3.25}$$

Whence and from (3.24) we obtain that

$$u_t = K[u] = -\vartheta grad\ H_\vartheta = -\eta grad\ H_\eta, \tag{3.26}$$

where

$$H_\vartheta = \int_0^{2\pi} uu_x^2 dx, \quad H_\eta = \int_0^{2\pi} u_x^2 dx. \tag{3.27}$$

The expression (3.23) can be rewritten in another useful form as

$$\lambda\vartheta\ grad\ \gamma(\lambda) = \eta\ grad\ \gamma(\lambda), \tag{3.28}$$

being in some sense equivalent [53] together with the equation (3.1) to the adjoint Lax type representation for the dynamical system

Theorem 3.2. *The dynamical system (3.4) on the functional manifold M is a compatible bi-Hamiltonian flow, possessing an infinite hierarchy of commuting functionally independent conservation laws, satisfying the fundamental gradient identity (3.28). The latter is equivalent together with the relationship (3.15) to the adjoint Lax type representation.*

An analysis of associated with (3.4) commuting flows $K_n := -\vartheta\ grad\ \gamma_n$, $n \in \mathbb{Z}$, shows an interesting property of their dispersionless. In particular, this means that all of them can not be treated effectively by means of the gradient-holonomic algorithm, namely, the asymptotic in $|\ \lambda\ | \to \infty$ solutions to the Lax equations

$$d\varphi/d\tau_n + K_n'^{*}\varphi = 0, \quad \varphi' \neq \varphi'^{+}, \tag{3.29}$$

where $du/d\tau_n = K_n[u],\ \tau_n \in \mathbb{R},\ n \in \mathbb{Z}$, do not give rise to explicit expressions, defining the associated hierarchy of conservation laws for the dynamical system

(3.4). Nonetheless, the corresponding hierarchy of dispersive commuting flows on M exists for (3.4), being associated with the trivial flow $du/dt_0 := 0$ on M. Namely, let $H_0 \in D(M)$ be a conservation law of (3.4), satisfying the kernel condition for the operator $\eta : T^*(M) \to T(M)$, that is

$$du/dt_0 = 0 := \eta \ grad \ H_0. \tag{3.30}$$

It is easy to find from (3.30) and (3.25), that $grad\ H_0 = [2(u_{xx})^{-1/2}]_{xx}$, whence

$$H_0 = 4 \int_0^{2\pi} \sqrt{u_{xx}} dx. \tag{3.31}$$

With the obtained invariant (3.31) there is associated the commuting with (3.4) flow

$$du/d\tau = -\vartheta \ grad \ H_0 = u_{xxx}(u_{xx})^{-3/2} := \tilde{K}[u], \tag{3.32}$$

$\tau \in \mathbb{R}$, which, as it is easy to see, already possesses a nontrivial dispersion. This means that the Lax equation

$$d\varphi/d\tau + \tilde{K}^{'*}\varphi = 0, \tag{3.33}$$

allows as $|\lambda| \to \infty$ an asymptotic solution $\varphi := \varphi(\lambda) \in T^*(M) \otimes \mathbb{C}$, where

$$\varphi(\lambda) \simeq exp(\int_{x_0}^{x} \sigma(y;\lambda)dy), \quad \sigma(x;\lambda) \simeq \sum_{j \in \mathbb{Z}_+} \sigma_j[u]\lambda^{-j} \tag{3.34}$$

with nontrivial functionals $H_j := \int_0^{2\pi} \sigma_j[u]dx, \quad j \in \mathbb{Z}_+$, being commuting conservation laws both of the dynamical system (3.32) and of our dynamical system (3.4).

The obtained results are important for further analytical studying Lax type integrability of the dynamical system (3.4) and finding, in particular, its wide class of special soliton like and quasi-periodic solutions. ==

4. The generalized de Rham-Hodge theory aspects of Delsarte-Lions type transmutation operators in multidimension

4.1. **Differential-geometric preliminaries.** A differential-geometric analysis of Delsarte-Darboux type transformations for differential operator expressions acting in a functional space $\mathcal{H} = L_2(\mathrm{T};H)$, where $\mathrm{T} = \mathbb{R}^2$ and $H := L_2(\mathbb{R}^2;\mathbb{C}^2)$, appears to have a deep relationship with the classical de Rham-Hodge theory [2, 63, 58, 59, 60, 61] devised in the midst of the past century for a set of commuting differential operators defined, in general on a smooth compact m-dimensional metric space M. Concerning our problem of describing the differential-geometric and spectral structure of Delsarte-Darboux type transmutations acting in $\mathcal{H}$, we preliminarily consider some backgrounds of the generalized de Rham-Hodge theory devised formerly by I.V. Skrypnik [58, 59, 60, 61] for studying special differential complexes. Consider a smooth metric space M being a suitably compactified form of the space $\mathbb{R}^m$, $m \in \mathbb{Z}_+$. Then one can define on $M_\mathrm{T} := \mathrm{T} \times M$ the standard Grassmann algebra $\Lambda(M_\mathrm{T};\mathcal{H})$ of differential forms on $\mathrm{T}\times M$ and consider a generalized external anti-differentiation operator $d_\mathcal{L} : \Lambda(M_\mathrm{T};\mathcal{H}) \to \Lambda(M_\mathrm{T};\mathcal{H})$ acting as follows: for any $\beta^{(k)} \in \Lambda^k(M_{\mathrm{T}};\mathcal{H})$, $k=\overline{0,m}$,

$$(4.1)\quad d_{\mathcal{L}}\beta^{(k)} := \sum_{j=1}^{2} dt_j \wedge \mathrm{L}_j(t;x|\partial)\beta^{(k)} + \sum_{i=1}^{m} dx_i \wedge \mathrm{A}_i(t;x;\partial)\beta^{(k)} \in \Lambda^{k+1}(M_{\mathrm{T}};\mathcal{H}),$$

where $\mathrm{A}_i \in C^2(\mathrm{T};\mathcal{L}(H))$, $i=\overline{1,m}$, are some differential operator mappings and

$$(4.2)\qquad \mathrm{L}_j(t;x|\partial) := \partial/\partial t_j - L_j(t;x|\partial)$$

$j=\overline{1,2}$, are suitably defined linear differential operators in $\mathcal{H}$, commuting with each other, that is

$$(4.3)\qquad [\mathrm{L}_1,\mathrm{L}_2]=0,\ [\mathrm{A}_k,\mathrm{A}_i]=0 \text{ and } [\mathrm{L}_j,\mathrm{A}_i]=0$$

for all $j=\overline{1,2}$ and $i,k=\overline{1,m}$.We will put, in general, that differential expressions

$$(4.4)\qquad L_j(t;x|\partial) := \sum_{|\alpha|=0}^{n_j(L)} a_\alpha^{(j)}(t;x)\frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

with coefficients $a_\alpha^{(j)} \in C^1(\mathrm{T};C^\infty(M;End\mathbb{C}^N))$, $|\alpha|=\overline{0,n_j(L)}$ $n_j^\alpha \in \mathbb{Z}_+$, $j=\overline{0,1}$, are some closed normal densely defined operators in the Hilbert space H for any $t\in\mathrm{T}$. It is easy to observe that the anti-differentiation of $d_{\mathcal{L}}$ defined by (**??**) is a generalization of the usual external anti-differentiation

$$(4.5)\qquad d=\sum_{j=1}^{m} dx_j \wedge \frac{\partial}{\partial x_j} + \sum_{s=1}^{2} dt_s \wedge \frac{\partial}{\partial t_s}$$

for which, evidently, commutation conditions

$$(4.6)\qquad [\frac{\partial}{\partial x_j};\frac{\partial}{\partial x_k}]=0,\quad [\frac{\partial}{\partial t_s};\frac{\partial}{\partial t_l}]=0,\ [\frac{\partial}{\partial x_j};\frac{\partial}{\partial t_s}]=0$$

hold for all $j,k=\overline{1,m}$ and $s,l=\overline{1,2}$. If now to substitute within (4.5) $\partial/\partial x_j \longrightarrow \mathrm{A}_j$, $\partial/\partial t_s \longrightarrow \mathrm{L}_s$, $j=\overline{1,m}$, $s=\overline{1,2}$, one gets the anti-differentiation

$$(4.7)\qquad d_{\mathcal{A}} := \sum_{j=1}^{m} dx_j \wedge \mathrm{A}_j(t;x|\partial) + \sum_{j=1}^{2} dt_s \wedge \mathrm{L}_s(t;x|\partial),$$

where the differential expressions $\mathrm{A}_j,\mathrm{L}_S:\mathcal{H}\longrightarrow\mathcal{H}$ for all $j,k=\overline{1,m}$ and $s,l=\overline{1,2}$, satisfy the commutation conditions $[\mathrm{A}_j,\mathrm{A}_k]=0$, $[\mathrm{L}_s,\mathrm{L}_s]=0$, $[\mathrm{A}_j,\mathrm{L}_s]=0$, then then operation (4.7) defines on $\Lambda(M_{\mathrm{T}};\mathcal{H})$ an anti-differential with respect to which the co-chain complex.

$$(4.8)\qquad \mathcal{H}\longrightarrow\Lambda^0(M_{\mathrm{T}};\mathcal{H})\xrightarrow{d_{\mathcal{A}}}\Lambda^1(M_{\mathrm{T}};\mathcal{H})\xrightarrow{d_{\mathcal{A}}}\dots\xrightarrow{d_{\mathcal{A}}}\Lambda^{\mathrm{m}+2}(M_{\mathrm{T}};\mathcal{H})\xrightarrow{d_{\mathcal{A}}}0$$

is evidently closed, that is $d_{\mathcal{A}}d_{\mathcal{A}}\equiv 0$. As the anti-differential (4.1) is a particular case of (4.7), we obtain that the corresponding to it co-chain complex (4.8) is closed too.

4.2. **Rigged Hilbert spaces of differential forms and the related the Hodge homologies.** Below we will follow ideas developed [58, 59, 60, 61]. A differential form $\beta \in \Lambda(M_{\mathrm{T}};\mathcal{H})$ will be called $d_{\mathcal{A}}$-closed if $d_{\mathcal{A}}\beta = 0$ and a form $\gamma \in \Lambda(M_{\mathrm{T}};\mathcal{H})$ will be called exact or $d_{\mathcal{A}}$-homological to zero if there exists on M_{T} such a form $\omega \in \Lambda(M_{\mathrm{T}};\mathcal{H})$ that $\gamma = d_{\mathcal{A}}\omega$.

Consider now the standard [63, 2] algebraic Hodge star-operation

$$* : \Lambda^{\mathrm{k}}(M_{\mathrm{T}};\mathcal{H}) \longrightarrow \Lambda^{\mathrm{m+2-k}}(M_{\mathrm{T}};\mathcal{H}), \tag{4.9}$$

$k = \overline{0, m+2}$, as follows: if $\beta \in \Lambda^{\mathrm{k}}(M_{\mathrm{T}};\mathcal{H})$, then the form $*\beta \in \Lambda^{\mathrm{m+2-k}}(M_{\mathrm{T}};\mathcal{H})$ is such that:

- $(m-k+2)$ - dimensional volume $|*\beta|$ of the form $*\beta$ equals k-dimensional volume $|\beta|$ of the form β;
- the $(m+2)$ -dimensional measure $\bar{\beta}^{\intercal} \wedge *\beta > 0$ under the fixed orientation on M_{T}.

Define also on the space $\Lambda(M_{\mathrm{T}};\mathcal{H})$ the following natural scalar product: for any $\beta, \gamma \in \Lambda^{\mathrm{k}}(M_{\mathrm{T}};\mathcal{H})$, $k = \overline{0, m}$,

$$(\beta, \gamma) := \int_{M_{\mathrm{T}}} \bar{\beta}^{\intercal} * \gamma. \tag{4.10}$$

Subject to the scalar product (4.10) one can naturally construct the corresponding Hilbert space

$$\mathcal{H}_{\Lambda}(M_{\mathrm{T}}) := \overset{m+2}{\underset{k=0}{\oplus}} \mathcal{H}_{\Lambda}^{k}(M_{\mathrm{T}}) \tag{4.11}$$

well suitable for our further consideration. Notice also here, that the Hodge star $*$-operation satisfies the following easily checkable property: for any $\beta, \gamma \in \mathcal{H}_{\Lambda}^{k}(M_{\mathrm{T}})$, $k = \overline{0, m}$,

$$(\beta, \gamma) = (*\beta, *\gamma), \tag{4.12}$$

that is the Hodge operation $* : \mathcal{H}_{\Lambda}(M_{\mathrm{T}}) \to \mathcal{H}_{\Lambda}(M_{\mathrm{T}})$ is unitary and its standard adjoint with respect to the scalar product (4.10) operation satisfies the condition $(*)' = (*)^{-1}$.

Denote by $d_{\mathcal{L}}^{'}$ the formally adjoint expression to the weak differential operation (4.1). By means of the operations $d'_{\mathcal{L}}$ and $d_{\mathcal{L}}$ in the $\mathcal{H}_{\Lambda}(M_{\mathrm{T}})$ one can naturally define [63, **?**, **?**, 58, 2] the generalized Laplace-Hodge operator $\Delta_{\mathcal{L}} : \mathcal{H}_{1}(M_{\mathrm{T}}) \longrightarrow \mathcal{H}_{1}(M_{\mathrm{T}})$ as

$$\Delta_{\mathcal{L}} = d'_{\mathcal{L}} d_{\mathcal{L}} + d'_{\mathcal{L}} d_{\mathcal{L}} \ . \tag{4.13}$$

Take a form $\beta \in \mathcal{H}_{\Lambda}(M_{\mathrm{T}})$ satisfying the equality

$$\Delta_{\mathcal{L}}\beta = 0. \tag{4.14}$$

Such a form is called [58, **?**] harmonic. One can also verify that a harmonic form $\beta \in \mathcal{H}_{\Lambda}(M_{\mathrm{T}})$ satisfies simultaneously the following two adjoint conditions:

$$d'_{\mathcal{L}}\beta = 0, \quad d_{\mathcal{L}}\beta = 0 \tag{4.15}$$

easily stemming from (4.13) and (4.14).

It is easy to check that the following differential operators in $\mathcal{H}_{\Lambda}(M_{\mathrm{T}})$

$$d_{\mathcal{L}}^{*} := *d'_{\mathcal{L}}(*)^{-1} \tag{4.16}$$

defines also a new external anti-differential operation in $\mathcal{H}_{\Lambda}(M_{\mathrm{T}})$.

Lemma 4.1. *The corresponding dual to (4.8) co-chain complex*

$$\mathcal{H} \longrightarrow \Lambda^0(M_{\mathrm{T}};\mathcal{H}) \xrightarrow{d^*_{\mathcal{L}}} \Lambda^1(M_{\mathrm{T}};\mathcal{H}) \xrightarrow{d^*_{\mathcal{L}}} ... \xrightarrow{d^*_{\mathcal{L}}} \Lambda^{m+2}(M_{\mathrm{T}};\mathcal{H}) \xrightarrow{d^*_{\mathcal{L}}} 0 \tag{4.17}$$

is exact.

Proof. A proof follows owing to the property $d^*_{\mathcal{L}} d^*_{\mathcal{L}} = 0$ holding due to the definition (4.16).▷

□

4.3. **The de Rham-Hodge type isomorphism and the structure of the Delsarte-Lions operators kernels.** Denote further by $\mathcal{H}^k_{\Lambda(\mathcal{L})}(M_{\mathrm{T}})$, $k = \overline{0, m+2}$, the cohomology groups of $d_{\mathcal{L}}$-closed and by $\mathcal{H}^k_{\Lambda(\mathcal{L}^*)}(M_{\mathrm{T}})$, $k = \overline{0, m+2}$, $k = \overline{0, m+2}$, the cohomology groups of $d^*_{\mathcal{L}}$-closed differential forms, respectively, and by $\mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L})}(M_{\mathrm{T}})$, $k = \overline{0, m+2}$, the abelian groups of harmonic differential forms from the Hilbert sub-spaces $\mathcal{H}^k_{\Lambda}(M_{\mathrm{T}})$, $k = \overline{0, m+2}$. Before formulating next results, define the standard Hilbert-Schmidt rigged chain [67, 68] of positive and negative Hilbert spaces of differential forms

$$\mathcal{H}^k_{\Lambda,+}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda,-}(M_{\mathrm{T}}), \tag{4.18}$$

the corresponding hereditary rigged chains of harmonic forms:

$$\mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),+}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L})}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),-}(M_{\mathrm{T}}) \tag{4.19}$$

and chains of cohomology groups:

$$\begin{aligned} \mathcal{H}^k_{\Lambda(\mathcal{L}),+}(M_{\mathrm{T}}) &\subset \mathcal{H}^k_{\Lambda(\mathcal{L})}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M_{\mathrm{T}}), \\ \mathcal{H}^k_{\Lambda(\mathcal{L}^*),+}(M_{\mathrm{T}}) &\subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*)}(M_{\mathrm{T}}) \subset \mathcal{H}^k_{\Lambda(\mathcal{L}^*),-}(M_{\mathrm{T}}) \end{aligned} \tag{4.20}$$

for all $k = \overline{0, m+2}$. Assume also that the Laplace-Hodge operator (4.13) is reduced upon the space $\mathcal{H}^0_{\Lambda}(M)al$. Now by reasoning similar to those in [63, 2] one can formulate a little generalized [59, 60, 61] de Rham-Hodge theorem.
The groups of harmonic forms $\mathcal{H}^k_{\Lambda,+}(M_{\mathrm{T}})$, $k = \overline{0, m+2}$, are, respectively, isomorphic to the homology groups $(H^k(M_{\mathrm{T}};\mathbb{C}))^{|\Sigma|}$, $k = \overline{0, m+2}$, where $H^k(M_{\mathrm{T}};\mathbb{C})$ is the k-th cohomology group of the manifold M_{T} with complex coefficients, a set $\Sigma \subset \mathbb{C}^p$, $p \in \mathbb{Z}_+$, is the set of suitable "spectral" parameters marking the linear space of independent $d^*_{\mathcal{L}}$-closed 0-form from $\mathcal{H}^0_{\Lambda(\mathcal{L}),-}(M_{\mathrm{T}})$ and, moreover, the following direct sum decompositions

$$\begin{aligned} \mathcal{H}^k_{\Lambda,+}(M_{\mathrm{T}}) &= \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),+}(M_{\mathrm{T}}) \oplus \Delta_{\mathcal{L}}\mathcal{H}^k_{\Lambda,+}(M_{\mathrm{T}}) \\ &= \mathcal{H}^k_{\Lambda(\mathcal{L}^*\mathcal{L}),+}(M_{\mathrm{T}}) \oplus d_{\mathcal{L}}\mathcal{H}^{k-1}_{\Lambda,+}(M_{\mathrm{T}}) \oplus d'_{\mathcal{L}}\mathcal{H}^{k+1}_{\Lambda,+}(M_{\mathrm{T}}) \end{aligned} \tag{4.21}$$

hold for any $k = \overline{0, m+2}$. Another variant of the statement similar to that above was formulated in [58, 59] and reads as the following generalized de Rham-Hodge theorem. The generalized cohomology groups $\mathcal{H}^k_{\Lambda(\mathcal{L}),+}(M_{\mathrm{T}})$, $k = \overline{0, m+2}$, are isomorphic, respectively, to the cohomology groups $(H^k(M_{\mathrm{T}};\mathbb{C}))^{|\Sigma|}$, $k = \overline{0, m+2}$. A proof of this theorem is based on some special sequence [58, 59, 60, 61, 62] of differential Lagrange type identities.▷ Define the following closed subspace

$$\mathcal{H}^*_0 := \{\varphi^{(0)}(\eta) \in \mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M_{\mathrm{T}}) : d^*_{\mathcal{L}}\varphi^{(0)}(\eta) = 0,\ \varphi^{(0)}(\eta)|_{\Gamma},\ \eta \in \Sigma\} \tag{4.22}$$

for some smooth $(m+1)$-dimensional hypersurface $\Gamma \subset M_{\mathrm{T}}$ and $\Sigma \subset (\sigma(L) \cap \bar{\sigma}(L)) \times \Sigma_\sigma \subset \mathbb{C}^p$, where $\mathcal{H}^0_{\Lambda(\mathcal{L}^*),-}(M_{\mathrm{T}})$ is, as above, a suitable Hilbert-Schmidt rigged[67, 68] zero-order cohomology group Hilbert space from the co-chain given by (4.20), $\sigma(L)$ and $\sigma(L^*)$ are, respectively, mutual generalized spectra of the sets of differential operators L and L^* in H at $t = 0 \in \mathrm{T}$. Thereby, the dimension $\dim\mathcal{H}_0^* = card\ \Sigma := |\Sigma|$ is assumed to be known. The next lemma first stated by I.V. Skrypnik [58, 59] is of fundamental meaning for a proof of Theorem 1.2.

Lemma 4.2. *There exists a set of differential $(k+1)$-forms $Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M_{\mathrm{T}};\mathbb{C})$, $k = \overline{0, m+2}$, and a set of k-forms $Z^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \in \Lambda^k(M_{\mathrm{T}};\mathbb{C})$, $k = \overline{0, m+2}$, parametrized by the set $\Sigma \ni \eta$,being semilinear in $(\varphi^{(0)}(\eta), \psi^{(k)}) \in \mathcal{H}_0^* \times \mathcal{H}_{\Lambda,+}^k(M_{\mathrm{T}})$, such that*

$$Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)}] = dZ^k[\phi^{(0)}(\eta\ ), \psi^{(k)}] \tag{4.23}$$

for all $k = \overline{0, m+2}$ and $\eta \in \Sigma$.

Proof. A proof is based on the following Lagrange type identity generalizing that of Part 1 and holding for any pair $(\varphi^0(\eta), \psi^(k)) \in \mathcal{H}_0^* \times \mathcal{H}_{\Lambda,+}^k(M_{\mathrm{T}})$:

$$\begin{aligned}
(4.24)0 &= < d_{\mathcal{L}}^*\phi^{(0)}(\eta), *(\psi^{(k)} \wedge \overline{\gamma}) >=< *d'_{\mathcal{L}}(*)^{-1}\varphi^{(0)}(\eta), *(\psi^{(k)} \wedge \overline{\gamma}) > \\
&= < *d'_{\mathcal{L}}(*)^{-1}\phi^{(0)}(x), \psi^{(k)} \wedge \overline{\gamma}) >= \\
&= < (*)^{-1}\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)} \wedge \overline{\gamma} > +Z^{(k+1)}[\psi^{(0)}(\eta), d_{\mathcal{L}}\psi(k)] \wedge \overline{\gamma} >= \\
(4.25) \quad &= < (*)_{-1}\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)} \wedge \overline{\gamma} > +dZ^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \wedge \gamma,
\end{aligned}$$

where $Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M_{\mathrm{T}};\mathbb{C})$, $k = \overline{0, m+2}$, and $Z^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \in \Lambda^k(M_{\mathrm{T}};\mathbb{C})$, $k = \overline{0, m+2}$, are some semilinear differential forms on M_{T} parametrized by a parameter $\lambda \in \Sigma$, and $\overline{\gamma} \in \Lambda^{m+1-k}(M_{\mathrm{T}};\mathbb{C})$ is arbitrary constant $(m+1-k)$-form. Thereby, the semilinear differential $(k+1)$-forms $Z^{(k+1)}[\varphi^{(0)}(\eta), d_{\mathcal{L}}\psi^{(k)}] \in \Lambda^{k+1}(M_{\mathrm{T}};\mathbb{C})$ and k-forms $Z^{(k)}[\varphi^{(0)}(\eta), \psi^{(k)}] \in \Lambda^k(M_{\mathrm{T}};\mathbb{C})$, $k = \overline{0, m+2}$, $\lambda \in \Sigma$, constructed above exactly constitute those searched for in the Lemma.$\triangleright$ $\square$

Based now on Lemma 1.3 one can construct the cohomology group isomorphism claimed in Theorem 1.2 formulated above. Namely, following [58, 59], let us take some singular simplicial [2, 5] complex $\mathcal{K}(M_{\mathrm{T}})$ of the compact metric space M_{T} and introduce a set of linear mappings $B_\lambda^{(k)} : \mathcal{H}_{\Lambda,+}^k M_{\mathrm{T}} \longrightarrow C_k(M_{\mathrm{T}};\mathbb{C})$, $k = \overline{0, m+2,}$ $\lambda \in \Sigma$, where $C_k(M_{\mathrm{T}};\mathbb{C})$, $k = \overline{0, m+2,}$ are free abelian groups over the field $\mathbb{C}$ generated, respectively, by all k-chains of singular simplexes $S^{(k)} \subset M_{\mathrm{T}}$, $k = \overline{0, m+2}$, from the simplicial complex $\mathcal{K}(M_{\mathrm{T}})$, as follows:

$$B_\lambda^{(k)}(\psi^{(k)}) := \sum_{S^{(k)} \in C_k(M_{\mathrm{T}};\mathbb{C}))} S^{(k)} \int_{S^{(k)}} Z^{(k)}[\varphi^{(0)}(\lambda), \psi^{(k)}] \tag{4.26}$$

with $\psi^{(k)} \in \mathcal{H}_{\Lambda,+}^k(M_{\mathrm{T}})$, $k = \overline{0, m+2}$. The following theorem [58, 59] based on mappings (4.26) holds.

Theorem 4.3. *The set of operators (4.26) parametrized by $\lambda \in \Sigma$ realizes the cohomology group isomorphism formulated in Theorem 1.2*

Proof. A proof of this theorem one can get passing over in (4.26) to the corresponding cohomology $\mathcal{H}_{\Lambda(\mathcal{L}),+}^k(M_{\mathrm{T}})$ and homology $H_k(M_{\mathrm{T}};\mathbb{C})$ groups of M_{T} for every $k = \overline{0, m+2}$. If one to take an element $\psi^{(k)} := \psi^{(k)}(\mu) \in \mathcal{H}_{\Lambda(\mathcal{L}),+}^k(M_{\mathrm{T}})$,

$k=\overline{0,m+2}$, solving the equation $d_{\mathcal{L}}\psi^{(k)}(\mu)=0$ with $\mu\in\Sigma_k$ being some set of the related "spectral" parameters marking elements of the subspace $\mathcal{H}^k_{\Lambda(\mathcal{L}),-}(M_{\mathrm{T}})$, then one finds easily from (4.26) and identity (4.23) that $dZ^{(k)}[\varphi^{(0)}(\lambda),\,\psi^{(k)}(\mu)]=0$ for all $(\lambda,\mu)\in\Sigma\times\Sigma_k$, $k=\overline{0,m+2}$. This, in particular, means due to the Poincare lemma [?, ?, ?] that there exist differential $(k-1)$-forms $\Omega^{(k-1)}[\varphi^{(0)}(\lambda),\psi^{(k)}(\mu]\in\Lambda^{k-1}(M;\mathbb{C})$, $k=\overline{0,m+2}$, such that

$$Z^{(k)}[\varphi^{(0)}(\lambda),\psi^{(k)}(\mu)]=d\Omega^{(k-1)}[\varphi^{(0)}(\lambda),\psi^{(k)}(\mu)] \tag{4.27}$$

for all pairs $(\varphi^{(0)}(\lambda),\psi^{(k)}(\mu))\in\mathcal{H}^*_0\times\mathcal{H}^k_{\Lambda(\mathcal{L}),+}(M_{\mathrm{T}})$ parametrized by $(\lambda,\mu)\in\Sigma\times\Sigma_k$, $k=\overline{0,m+2}$. As a result of passing on the right hand-side of (4.26) to the homology groups $H_k(M_{\mathrm{T}};\mathbb{C})$, $k=\overline{0,m+2}$, one gets due to the standard Stokes theorem [?, ?, ?] that the mappings

$$B^{(k)}_\lambda:\mathcal{H}^k_{\Lambda(\mathcal{L}),+}(M_{\mathrm{T}})\longrightarrow H_k(M_{\mathrm{T}};\mathbb{C}) \tag{4.28}$$

are isomorphisms for every $k=\overline{0,m+2}$ and $\lambda\in\Sigma$. Making further use of the Poincare duality [63, 2] between the homology groups $H_k(M_{\mathrm{T}};\mathbb{C})$, $k=\overline{0,m+2}$, and the cohomology groups $H^k(M;\mathbb{C})$, $k=\overline{0,m+2}$, respectively, one obtains finally the statement claimed in Theorem 1.4.▷

□

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