RESEARCH PLAN OF BORISLAV GAJIĆ

Theory of integrable systems has been intensively developed since late 1960's. Appearance of Lax representation gave an important impulse to the theory. The Lax representation is the first step in algebro-geometric integration procedure. Complete integration procedure is based on the Baker-Akhiezer functions, developed in the middle od 1970's.

One of the important problems in the theory of integrable systems is to find a discrete counterpart for a given integrable system. In the recent period a great progress has been made in a study of discrete integrable systems, after Veselov, Moser, Suris, Bobenko and others.

The discrete time Lagrange mechanics has been studied for the first time by Veselov and Moser, and further developed by Suris and Bobenko. Discrete time Euler-Lagrange equations can be derived from the discrete time variational principle. In the papers [3] and [4] (see the list given below), theory of discrete time Lagrange mechanics on Lie group G is developed. Equations of the motion on $G \times G$, in both left and right trivialization are given. In the case when Lagrange function is invariant with respect to the left or right multiplication by elements of some subgroup, it was shown, using discrete Lagrange reduction procedure, that reduced equations lives on the certain semidirect product. By applying that theory to the rigid body motion, integrable discretizations where found for several integrable cases: the Euler case, the Lagrange case, the case of the rigid body in quadratic potential and others (see [1-4]). However, rigid body motion provides many integrable cases, including higher-dimensional generalizations.

We plan to apply the theory to other integrable cases of rigid body dynamics, and to find their discrete counterparts. Also, we plan to find their discrete Lax representations. For example we shell consider n-dimensional generalizations of the Lagrange top on semidirect product $so(n) \times so(n)$.

The three-dimensional Lagrange case is one of the classical integrable cases of motion of a heavy rigid body fixed at a point. Equations of the motion are Hamiltonian in standard Poisson structure on Lie algebra $e(3) = \mathbb{R}^3 \times so(3)$. Simple Lax representation is given by Ratiu and van Moerbeke. Discrete counterpart in both, moving and rest frame formulation, was studied in [3] by Bobenko and Suris. In the moving frame formulation, equations of motion are:

$$(1) \qquad M_{k+1} = W_k^{-1} M_k W_k + \epsilon A \wedge P_{k+1}, \quad P_{k+1} = W_k P_k,$$

where W_k correspond to the angular velocity, M_k to the angular momentum, P_k to the unit vertical vector, and A is a constant vector. The map $(M_k, P_k) \mapsto (M_{k+1}, P_{k+1})$, defined by (1), is Poisson with respect to the standard Lie-Poisson structure on $so(3)$. Lax representation of the (1) was also given in [3].

In the rest frame formulation equations are:

$$(2) \quad m_{k+1} = m_k + \epsilon a_k \wedge p, \quad a_{k+1} = (I + (\epsilon/2) m_k + 1) a_k (I - (\epsilon/2) m_k + 1)^{-1},$$

The map $(m_k, a_k) \mapsto (m_{k+1}, a_{k+1})$ is Poisson with respect to the standard Lie-Poisson structure on $so(3)$.

One of the natural higher-dimensional generalizations of the Lagrange top is on the Lie algebra $e(n) = \mathbb{R}^n \times so(n)$. Integrability was shown by Belyayev. Certain integrable discretization, (in moving and rest formulations) was given by Suris in [1]. Discrete Lax representations are constructed.

However, using isomorphism between $\mathbb{R}^3$ and $so(3)$ one can consider three dimensional Lagrange top as a Hamiltonian system on semidirect product $so(3) \times so(3)$. That serves motivation for the other higher-dimensional generalization of the Lagrange top on semidirect product $so(n) \times so(n)$. Corresponding equations of the motion and Lax representations were constructed by Ratiu.

In [7] the algebro-geometric integration procedure for the Lagrange bitop on $so(4) \times so(4)$ is given. It is based on deep facts from the geometry of Prym varieties of double coverings of hyperelliptic curves: Mumford's relation and Mumford-Dalalyan theory. Systems of Hess-Appel'rot type on $so(n) \times so(n)$ are introduced in [5]. They are certain perturbation of Lagrange top.

Our plan is to find integrable discretization of n-dimensional Lagrange top on $so(n) \times so(n)$ and construction of discrete Lax representation. The goal of our investigations is to get explicit formulae in theta-functions. This project is based on our previous papers:

[1] Yu.B. Suris: *Integrable discretizations of some cases of the rigid body dynamics*, J. Nonlin. Math. Phys., 2001, 8, p. 534-560. nlin.SI/0105012

[2] Yu.B. Suris: *The motion of a rigid body in a quadratic potential: an integrable discretization*, Intern. Math. Research Notices, 2000, 12, p. 643-663, solv-int/9909009

[3] A.I. Bobenko, Yu.B. Suris: *Discrete time Lagrangian mechanics on Lie groups, with an application to the Lagrange top*, Commun. Math. Phys., 1999, 204, p. 147-188, solv-int/9810018

[4] A.I. Bobenko, Yu.B. Suris: *Discrete Lagrangian reduction discrete Euler-Poison equations and semidirect product*, Lett. Math. Phys., 1999, 49, p. 79-93

[5] V. Dragović, B. Gajić: *Systems of Hess-Appel'rot type*, to appear in Commun. Math. Phys., (2006).

[6] V. Dragović, B. Gajić: *Matrix Lax polynomials, geometry of Prym varieties and systems of Hess-Appel'rot type*, Letters in Math. Phys, to appear (2006).

[7] V. Dragović, B. Gajić: *The Lagrange bitop on* $so(4) \times so(4)$ *and geometry of the Prym varieties*, American Journal of Mathematics 126 (2004), no 5, 981-1004; math-ph/ 0201036

[8] V. Dragović, B. Gajić: *L-A pair for the Hess-Apel'rot system and new integrable case in* $so(4) \times so(4)$, Proceedings of Royal society of Edinburgh A, 131 (2001), no. 4, 845-855; math-ph/9911047