CASE FOR SUPPORT
Hydrodynamic reductions of dispersionless Veselov-Novikov equation.

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1. PROGRAMME.

Some of most important developments in modern nonlinear optics are concerning with the role of nonlocality in the propagation of stable laser beams (usually called spatial solitons) in 3(or 2+1)-dimensions[1–5].

It should be emphasized that spatial nonlocality is ubiquitous. It is observed in many experimental contexts (laser beams propagation in nematic liquid crystals, plasma physics, Bose-Einstein condensation, see e.g. [4] and references therein). Nevertheless, theoretical study of nonlocal e_ects is in fact at the beginning. In particular, the study of integrable 2+1-dimensional models of laser beams propagation in nonlinear and nonlocal media is completely open except for the highly nonlocal regime, investigated by Snyder and Mitchell [1]. In this very special limit the problem of integrability is trivial one since nonlinear Schroedinger equation collapses onto the quantum harmonic oscillator equation.

In my recent papers in collaboration with Prof. Konopelchenko [6–8] I investigated possible existence of new integrable regimes in the propagation of a paraxial laser beam in high frequency limit for a general class of nonlinear nonlocal media (liquid crystals, several liquid and solid polar media) obeying the Cole-Cole dispersion law [9]. Results mentioned in the following and their developments were reported at the SPIE (The International Society for Optical Engineering) International Conference on Optics and Optoelectronics (Warsaw 29 August - 02 September 2005) and will be published on Proceedings of SPIE vol. 5949.
In particular we have shown the existence of a one-to-one correspondence among different degrees of nonlocality and the equations of dispersionless Veselov-Novikov (dVN) hierarchy, provided by the system

$$\begin{aligned} &S_x^2 + S_y^2 = 4u \\ &S_z = \varphi\left(S_x, S_y, x, y, z\right) \end{aligned} \qquad (1)$$

where x, y and z are the spatial coordinates, S the phase of the electric field, u is the refractive index and the function φ is a odd degree polynomial (corresponding to the nonlocality degree) in S_x and S_y. Compatibility condition of the system (1) is equivalent to the dVN hierarchy for the refractive index u. In particular, dVN equation

$$
\begin{aligned}
&u_z = (V_1 u)_x + (V_2 u)_y \\
&V_{1x} - V_{2y} = -3u_x \\
&V_{1y} + V_{2x} = 3u_y
\end{aligned} \tag{2}
$$

is associated with the third nonlocality degree (third degree polynomial).

The dVN hierarchy characterizes both the phase of the electric field and the refractive index. It should be stressed that the arising of dVN hierarchy in the context of nonlinear optics is new and it could open the way to a novel interesting phenomenology of the interplay between nonlinearity and nonlocality.
A complete description of this physical system requires the combination of dVN equation with the Poynting vector conservation law for a nonlinear medium such that the intensity law $I = I(u)$ is verified, where I is the intensity of the electric field.

Then, the set of equations so obtained (dVN equation + Poynting vector conservation + intensity law) is highly over-determined and the compatibility condition implies nontrivial restrictions on the form of intensity law. In the reference [8] we have shown that a subclass of solutions of some hydrodynamic type reductions of dVN equation provides us with compatible intensity laws. At the moment, hydrodynamic reduction method seems to be the only fruitful approach to treat the present system.
Recently, Ferapontov and Khusnutdinova [10–12] proposed the method of hydrodynamic reductions as very a powerful approach to the problem of integrability and classifications of multidimensional dispersionless PDEs.
Several 2+1-dimensional cases, such as dKP, (modified) Boyer-Finley have been discussed explicitly.
My discussions with Prof. Ferapontov have convinced me that his ideology can be applied effciently to dVN equation in order to calculate all hydrodynamic reductions and possibly all (however a very large class of) compatible intensity laws.
Although dVN equation falls in the class considered by Ferapontov and Khusnutdinova, it is a highly nontrivial case which was not studied yet. Once it will be done, a direct comparison between exact physical predictions so obtained and real optical responses of nonlocal nonlinear media will be possible.

2. AIM OF THE VISIT.

Being Prof. Ferapontov one of leading experts on multidimensional dispersionless integrable systems, the visit to him at Loughborough University is crucial in order this project to be successful.
In consideration that some preliminary results about hydrodynamic-type reductions of dVN equation was already obtained jointly to Proff. Bogdanov and Konopelchenko using the so-called symmetry constraints approach [13] and that the Ferapontov method for 2+1-dimensional quasilinear systems is well established, I estimate an eight-days visit to be enough to get the intended results.

Best period for the visit would be the week 21-28 November 2005, in coincidence with the "Integrability day in Loughborough", a half-day workshop which will be held at 26 November 2005.

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