Discrete Curvature Flows and Their Applications

Ivan Izmestiev

Application for an ESF Exchange Grant

1 Aim of the visit and research project

I plan to spend three weeks from 6th till 25th of June 2005 at the Mathematical Physics Sector of SISSA.

My research area is discrete differential geometry. In this subject one studies discrete analogs of differential geometric constructions. This idea is of course not new; many researchers have been interested in the interplay between smooth and discrete. One of the most prominent examples is Regge's calculus developed by Regge in his paper "General relativity without coordinates" in 1961. In the last years the idea of synthesis of discrete and differential geometry attracts much attention in analysis, geometry, and topology. Not the last role is played by applications in computer graphics, where one often has to picture smooth objects using discrete ones. One of the recent results in discrete differential geometry is application of circle packings to the approximation of conformal mappings. The idea was first suggested by Thurston in 70's. In the last 15 years there appeared various results on existence and uniqueness of a circle packing with a given combinatorics or a circle pattern characterized by some geometric quantities. Finally this approach found applications in computer science due to works of Stephenson and Schröder. An equally active field adjacent to discrete differential geometry is the theory of discrete integrable systems.

Specific subjects I am dealing with are: Alexandrov's theorem, discrete Willmore energy, and discrete curvature flows.

Alexandrov's theorem states that a euclidean metric with convex conic singularities on a sphere can be realized in a unique up to ambient isometry way by the boundary of a convex polytope in the euclidean space. This can be considered as the solution to a discrete version of Weyl's embedding problem: Can every Riemannian metric of positive curvature on a sphere be induced by some embedding of the sphere into the Euclidean space? Later Pogorelov extended Alexandrov's theorem, solving in particular Weyl's problem. He approximated a Riemannian metric by polyhedral ones. Though proved more than 60 years ago, Alexandrov's theorem remains misterious in some respects. In particular, a constructive proof is missing; the original proof is the only known one and presents a pure existence and uniqueness argument. Alexandrov himself has posed the question of finding a functional on some parameter space such that the critical points of the functional correspond to realizations of the metric by boundaries of polytopes. If, besides that, the functional is convex and satisfies together with its domain some simple additional assumptions, one has the existence and uniqueness of a realization. There are some approaches known to finding a variational proof of this kind, e.g. the one suggested by Blaschke and Herglotz (actually, in relation to Weyl's problem).

A discrete conformal energy of a simplicial surface analogous to Willmore energy of a smooth embedded surface was proposed by Bobenko in his recent preprint. It is defined through the sum of the angles between the circumscribed circles of adjacent triangles. It can be shown that the discrete conformal energy indeed converges to the Willmore energy, if one chooses the approximating simplicial surfaces to be Delaunay triangulated. From Bryant's work it is known that the Willmore energy is quantized in critical points by factor 4π. There is evidence that a similar phenomenon occurs with the discrete conformal energy, though with the factor 2π. Bobenko has shown that a simplicial surface has energy 0 if and only if it is a convex inscribed simplicial polytope. It was already known to Steinitz that there are triangulations that cannot be represented by an inscribed convex polytope, and hence their minimal energy is positive. This is closely related to the question which triangulations can be realized as Delaunay trianglulations. On the other hand, hyperbolic geometry becomes involved in the important special case when all of the vertices of the triangulation lie on a sphere. Consider the hyperbolic space modelled in the interior of this sphere. Then the angles between circumferences of triangles are equal to the angles between the faces of an ideal hyperbolic simplicial surface. When looking for the minimum of the discrete conformal energy, one comes to consider negatively oriented ideal tetrahedra. At the same time such tetrahedra have been some sort of nuisance in the study of ideal triangulations of hyperbolic manifolds, as explained in works of Petronio. Thus, the progress in investigating exrema of discrete conformal energy can be connected with studying moduli spaces of hyperbolic polytopes and structure of ideal triangulations involving negatively oriented triangles.

In recent papers and preprints of Chow, Luo, and Glickenstein discrete versions of Ricci and Yamabe flows were introduced. The classical Ricci flow introduced by Hamilton in 80's gained in popularity after works of Perelman on Thurston's geometrization conjecture. The Yamabe problem asks if there is a constant scalar curvature metric in the conformal equivalence class of any Riemannian metric. Its affirmative solution in 60's – 80's was considered as a milestone in application of non-linear PDEs in geometry. The idea of discretization of a curvature flow consists in the following. Represent a manifold as a result of gluing of geometrical simplices (either euclidean or spherical or hyperbolic). At the faces of codimension 2 there arise either angle defects or excesses. This allows to define some discrete analog of curvature. So done, one lets the edges (or angles) of simplices to expand or contract according to a system of ordinary differential equations involving the curvature. The hope is that the flow converges to triangulated manifold of constant curvature (or of constant discrete curvature). With the help of the combinatorial Ricci flow on surfaces Chow and Luo obtained a new proof of the Thurston's theorem on existence of a circle packing with prescribed combinatorics. We hope to apply discrete curvature flows to studying the problems mentioned above, Alexandrov's theorem and discrete conformal energy.

2 Organizational details

Host

Professor Boris Dubrovin
SISSA/ISAS
via Beirut 4
34014 Trieste, Italy
E-mail: dubrovin@sissa.it

Starting date of the visit: 6th June 2005

Estimated travel costs: 250 €