Algebraic solution of the Schlesinger system

Tamara Grava

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The visit was carried out to continue a scientific collaboration started with Dr. M. Mazzocco. During my stay in Manchester we investigated the conditions for which particular solutions $A_k(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of the 3×3 Schlesinger system

$$\frac{\partial}{\partial \lambda_j} A_k = \frac{[A_j, A_k]}{\lambda_j - \lambda_k}, \quad j \neq k, \quad \frac{\partial}{\partial \lambda_k} A_k = -\sum_{\substack{j \neq k \\ j=1}}^{2m+1} \frac{[A_j, A_k]}{\lambda_j - \lambda_k}, \quad j,k = 1,2,3, \quad (1)$$

are algebraic. In the above the matrices $A_k \in Mat(3, \mathbb{C})$, are traceless with eigenvalues equal to $-2/3, 0, 2/3$, the point $\lambda_4 = \infty$ and $A_4 = -A_1 - A_2 - A_3$. The search for algebraic solutions of the Schelsinger system has been so far carried out only for 2×2 matrices A_k with four singular points. This problem is equivalent to obtain algebraic solutions of the Painlevé VI equation.

The particular solutions of the 3×3 Schlesinger system we are considering are obtained from the solution of a 3×3 matrix Riemann-Hilbert problem with four singular points and with quasi-permutation monodromy matrices of the form

$$M_1 = \begin{pmatrix} 0 & 0 & c_1 \\ \frac{c_2}{c_1} & 0 & 0 \\ 0 & \frac{1}{c_2} & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & \frac{c_1 d_1}{c_2} & 0 \\ 0 & 0 & c_2 d_2 \\ \frac{1}{c_1 d_1 d_2} & 0 & 0 \end{pmatrix},$$
$$M_3 = \begin{pmatrix} 0 & 0 & d_1 d_2 \\ \frac{1}{d_1} & 0 & 0 \\ 0 & \frac{1}{d_2} & 0 \end{pmatrix}, \quad M_\infty = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (2)$$

where c_1, c_2, d_1, d_2 are non-zero complex constants.

The matrices A_k can be written explicitly in terms of the θ-function on a 3-sheeted

Riemann surfaces of genus 2 with period matrix Π [1]

$$(A_k)_{ss} = (\lambda_0 - \lambda_k)^2 \frac{\partial}{\partial \lambda_k} \left(\sum_{l=1}^{2} \frac{\partial}{\partial z_l} \log \theta \begin{bmatrix} \boldsymbol{\delta} \\ \boldsymbol{\epsilon} \end{bmatrix} (\mathbf{0}; \Pi) \left. \frac{dv_l(P)}{dx(P)} \right|_{P=P_0^{(s)}} \right), \quad s = 1, 2, 3,$$

$$(A_k)_{rs} = \mathrm{sign}(s-r)(-1)^{(s-r)} \frac{\sqrt{3}i}{9} (\lambda_0 - \lambda_k)^2 \frac{\partial}{\partial \lambda_k} \left[\left(\sum_{l=1}^{3} \frac{(-1)^l}{\lambda_0 - \lambda_l} \right) \frac{\theta \begin{bmatrix} \boldsymbol{\delta} \\ \boldsymbol{\epsilon} \end{bmatrix} (\boldsymbol{e}_{sr}; \Pi)}{\theta (\boldsymbol{e}_{sr}; \Pi)} \frac{\theta(\mathbf{0}; \Pi)}{\theta \begin{bmatrix} \boldsymbol{\delta} \\ \boldsymbol{\epsilon} \end{bmatrix} (\mathbf{0}; \Pi)} \right]$$

where $s \neq r$, $r, s = 1, 2, 3$, the vectors $\boldsymbol{e}_{sr}$ do not depend on the characteristics $\boldsymbol{\epsilon}$, $\boldsymbol{\delta}$ and $d\boldsymbol{v} = (dv_1, dv_2)$ are the normalized holomorphic differentials. The point λ_0 in the above formula is the base point and the matrix A_4 is recovered from the relation $A_4 = -A_1 - A_2 - A_3$.

The characteristics $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2)$ and $\boldsymbol{\delta} = (\delta_1, \delta_2)$ in the θ-function are related to the entries of the monodromy matrices by the equations

$$\epsilon_i = \frac{1}{2\pi i} \log c_i, \quad \delta_i = \frac{1}{2\pi i} \log d_i, \quad i = 1, 2.$$

Given such an explicit formula for the solution of the Schlesinger system, we prove that

1. the structure of the nonlinear monodromy is induced by the action of the subgroup $\Gamma_0(3)$ of the modular group on the parameters (ϵ_1, ϵ_2) and (δ_1, δ_2). This property resembles the property of the Picard solution of the Painlevé VI whose nonlinear monodromy is described by the action of $\Gamma(2)$ on the parameters [2].

(2) The family of solutions with rational parameters are algebraic solutions of the 3×3 Schlesinger system. Furthermore we show that in this case the structure of the nonlinear monodromy is also induced by the action of the symplectic group $Sp(4, \mathbb{Z})$ on the parameters (ϵ_1, ϵ_2) and (δ_1, δ_2).

Further developments of our research are

1. obtain an explicit parameterization of some of the algebraic solution of the 3×3 Schlesinger system.

2. compare our results with the results of Stefanov [3] about the finite orbits of the braid group action on Stokes matrices.

The development of the above two points will complete our work and all the results will be published in a joint paper.

References

[1] V.Enolski, T.Grava, Singular Z_N-curves and the Riemann-Hilbert problem. Int. Math. Res. Not. 2004, no. 32, 1619-1683.

[2] M. Mazzocco, Picard and Chazy solutions to the Painlevé VI equation. Math. Ann. 321 (2001), no. 1, 157-195.

[3] A. Stefanov, Finite orbits of the braid group action on a set of reflections, http://xxx.lanl.gov/math-ph/0409026