Algebraic solution of the Schlesinger system

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The visit was carried out to continue a scientific collaboration started with Dr. M. Mazzocco. During my stay in Manchester we investigated the conditions for which a 3×3 Schlesinger system of the form

$$\frac{\partial}{\partial a_j}A_k = \frac{[A_j, A_k]}{a_j - a_k}, \quad j \neq k, \quad \frac{\partial}{\partial a_k}A_k = -\sum_{\substack{j\neq k\\ j=1}}^{2m+1} \frac{[A_j, A_k]}{a_j - a_k}, \quad j,k = 1,2,3,4. \quad (1)$$

has algebraic solutions.

A Fuchsian system is described by the following differential equation with meromorphic coefficients

$$\frac{d}{dz}y(z) = \sum_{i=1}^{m} \frac{A_i}{z - a_i} y(z), \quad z \in \mathbb{CP}^1,$$

where y is an N-dimensional vector and the $N \times N$ matrices A_i, $i = 1, \dots m$, do not depend on z. The fundamental solution $Y(z)$ of the Fuchsian system has in general singularities of branching type near the points $z = a_i$, $i = 1, \dots, m$. The analytic continuation of the solution around one of its singularity is described by the monodromy matrices M_i,

$$Y(z)|_{(z-a_i)\to(z-ai)e^{2\pi i}} = Y(z)M_i, \quad M_i \in GL(N, \mathbb{C}), \quad i = 1, \dots m.$$

Riemann was the first to point out the problem of reconstruction of the Fuchsian system starting from its monodromy matrices and today such a problem is called Riemann-Hilbert (RH) problem. Explicit solutions of the Riemann-Hilbert problem have been obtained by various authors when the monodromy matrices belong to $GL(2, \mathbb{C})$. When the monodromy matrices belong to $GL(N, \mathbb{C})$, $N \geq 2$, and are quasi-permutation (that is each row and column has only one non-zero element), Korotkin has introduced a formal procedure for solving the Riemann-Hilbert problem.

From the solution of the Riemann-Hilbert problem it is possible to obtain a particular solution of the $N \times N$ Schlesinger system. The Schlesinger system describes the deformation of the matrices

$$A_i = A_i(a_1, \ldots, a_m | M_1, \ldots, M_m)$$

in the space of parameters $\{a_1, \ldots, a_m\}$ in such a way that the monodromy of the corresponding Fuchsian system remains constant. For this reason the Schlesinger equations are also called isomonodromic deformation equations. For $N = 2$ and $m = 4$, the Schlesinger equations are equivalent to the Painlevè VI equation.

In the paper 8 in the publication list, we have studied in detail the solution of a 3×3 RH problem with four singular points and with quasi-permutation monodromy matrices. The solution of the problem is obtained in terms of theta-functions defined on the Jacobian variety of a genus two Riemann surface that is a 3-sheeted covering of the complex plane. The total number of parameters appearing in the problem is four, namely the four-dimensional characteristics appearing in the theta-function. Then we have derived the solution of the corresponding Schelsinger system.

In this research project we plan to study the algebraic solutions of this Schlesinger system.

After applying the conformal transformation that maps the four singular points to 0, 1, t and ∞ we plan to study the analytic continuation of the solutions of the Schlesinger system induced by the generators of the fundamental group $\pi_1(\mathbb{CP}^1 \backslash \{0, 1, \infty\}, t)$ with base point t. We plan to show that the structure of the nonlinear monodromy is induced by the action of the subgroup $\Gamma_0(3)$ of the modular group analogously to the Picard solution of the Painlevé VI equation whose nonlinear monodromy is described by the action of $\Gamma(2)$ on the space of parameters. The corresponding algebraic solutions are given by the finite orbits of this group action.