Discrete isomonodromic Green's function on quad-graphs

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Based on:

A. Bobenko, Ch. Mercat, Yu. Suris.

Linear and nonlinear theories of discrete analytic functions. Integrable structure and isomonodromic Green's function.

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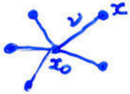
Generalization for graphs

$G=(V,E)$ – graph with set of vertices V and set of edges E.

$\nu: E\to\mathbb{R}_+$ – weights

Laplace operator acts on $u: V\to\mathbb{R}$,

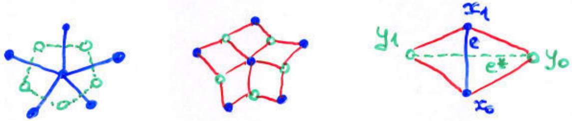
$$(\Delta u)(x_0)=\sum_{x\sim x_0}\nu(x_0,x)\big(u(x)-u(x_0)\big)$$



If G planar:

$G^*=(V^*,E^*)$ – dual graph

D – quad-graph, with vertices $V(D)=V\sqcup V^*$ and with quadrilateral faces, each having a dual pair of edges $e\in E$, $e^*\in E^*$ as diagonals



Weights on E^*: $\nu(e^*)=\frac{1}{\nu(e)}$

Discrete holomorphic functions $w: V(D)=V\sqcup V^*\to\mathbb{C}$ satisfy <u>discrete CR eqs.</u> on all (positively oriented) quadrilaterals of D:

$$\boxed{\frac{w(y_1)-w(y_0)}{w(x_1)-w(x_0)}=i\nu(x_0,x_1)=-\frac{1}{i\nu(y_0,y_1)}}$$

Th. (i) $w: V(D) \to \mathbb{C}$ discrete holomorphic $\Rightarrow$

$w|_{V(G)}$, $w|_{V(G^*)}$ discrete harmonic

(ii) $u: V(G) \to \mathbb{C}$ discrete harmonic $\Rightarrow$

$\Rightarrow \exists!$ (up to an additive constant) extension to a discrete holomorphic $w: V(D) \to \mathbb{C}$ with $w|_{V(G)} = u$.

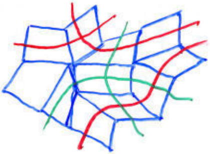
Rhombic quad-graphs

Quad-graph D: cell decomposition of a surface ($\mathbb{R}^2$) with all quadrilateral faces;

Rhombic quad-graph: all faces – rhombi [Duffin'68]

$\alpha_i \in \mathbb{C}$, $|\alpha_i| = 1$

Combinatorial characterization of quad-graphs admitting a rhombic realization: consider strips.



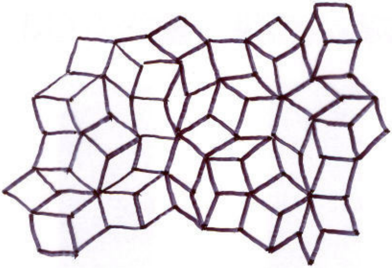
Rhombic realizations exist iff

(i) no strip crosses itself or is periodic

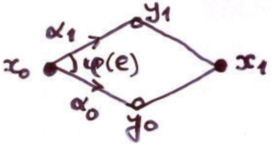
(ii) any two distinct strips cross at most once

[Kenyon, Schlenker '04]

Quasicrystallic rhombic quad-graph: the number of different edge slopes is finite: $\alpha_1, \alpha_2, \dots, \alpha_d \in S^1$. Examples: double-periodic graphs, Penrose tilings, etc.



Special weights on rhombic quad-graphs

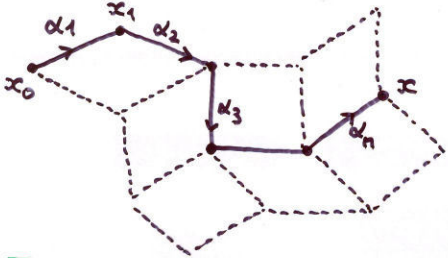


$$\boxed{i\nu(e) = i \tan \frac{\varphi(e)}{2} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\alpha_1 - \alpha_0}{\alpha_1 + \alpha_0}}$$

dCR:

$$\boxed{\frac{w(y_1) - w(y_0)}{w(x_1) - w(x_0)} = \frac{\alpha_1 - \alpha_0}{\alpha_1 + \alpha_0} = \frac{y_1 - y_0}{x_1 - x_0}}$$

Important example of discrete holomorphic function on a rhombic quad-graph : exponential function [Duffin '68]

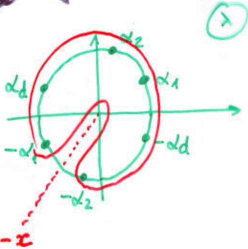


$$\boxed{e(x,\lambda) = \prod_{k=1}^{n} \frac{\lambda+\alpha_k}{\lambda-\alpha_k}}$$

What is so special about special weights?
[Kenyon, Inventiones '02] : explicit formula for the Green' function $g_{x_0}(x)$

(defined by (i) $\Delta_x g_{x_0}(x) = \delta_{x_0}(x)$
(ii) $g_{x_0}(x) \underset{x\to\infty}{\sim} \log|x-x_0|$)

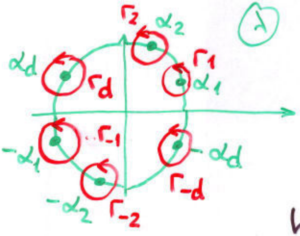
$$\boxed{g_{x_0}(x) = \oint_{\Gamma} \frac{1}{2\pi i} \frac{\log\lambda}{\lambda} e(x,\lambda)\,d\lambda}$$



[BMS]:

- $\boxed{\text{special weights} \Leftrightarrow \text{dCR integrable}}$,
- discrete Green's function for special weights is isomonodromic
- Th. On a quasicrystallic rhombic quad-graph with special weights, discrete exponential functions form a basis in the space of discrete holomorphic functions:

$$\boxed{w(x) = \frac{1}{2\pi i} \oint_{\Gamma} \rho(\lambda)\, e(x,\lambda)\, d\lambda}$$

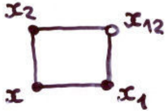


$\Gamma = \Gamma_1 \sqcup \ldots \sqcup \Gamma_d \sqcup \Gamma_{-1} \sqcup \ldots \sqcup \Gamma_{-d}$

The spectral function $\rho(\lambda)$ is defined and holomorphic on $U_1 \sqcup \ldots \sqcup U_d \sqcup U_{-1} \sqcup \ldots \sqcup U_{-d}$, where $U_{\pm k}$ are small neighborhoods of $\pm\alpha_k$, and the integration path $\Gamma = \Gamma_1 \sqcup \ldots \sqcup \Gamma_{-d}$ consists of small circles $\Gamma_{\pm k} \subset U_{\pm k}$ around $\pm\alpha_k$.

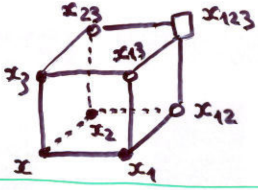
Integrability of discrete 2D equations as their 3D consistency.

2D equation:



$$Q(x, x_1, x_2, x_{12}) = 0$$

3D consistency:

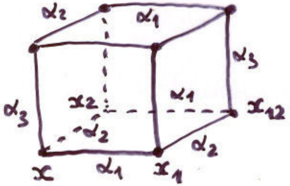


$$\boxed{3 \text{ values for } x_{123} \text{ coincide } \forall x, x_1, x_2, x_3}$$

3D consistency $\Longrightarrow$ Lax representation [Bobenko, Suris '2002, Nijhoff '2002]

3D consistency $\Longrightarrow$ Bäcklund transformations

Th. dCR with special weights are 3D consistent



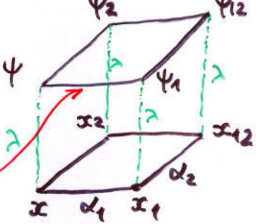
$$\frac{x_{12} - x}{x_2 - x_1} = \frac{\alpha_2 + \alpha_1}{\alpha_2 - \alpha_1}$$

$$\boxed{\frac{x_{ij} - x}{x_j - x_i} = \frac{\alpha_j + \alpha_i}{\alpha_j - \alpha_i}}$$

Conversely, dCR equations on a quad-graph D are integrable (3D consistent), only if the weights $i\nu$ come from a parallelogram realization of D, as quotients of diagonals of the parallelograms. Have $\nu > 0 \Leftrightarrow$ parallelograms are rhombi.

Lax representation

$$\psi_1 = \underbrace{L(x_1, x; \alpha_1, \lambda)} \cdot \psi$$



$$L(x_1, x; \alpha_1, \lambda) =$$

$$= \begin{pmatrix} \lambda + \alpha_1 & -2\alpha_1(x + x_1) \\ 0 & \lambda - \alpha_1 \end{pmatrix}$$

triangular (reflects the linearity of eqs)

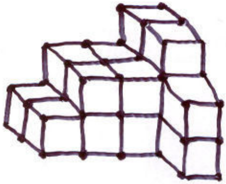
$$L(x_{12}, x_1; \alpha_2, \lambda)\, L(x_1, x; \alpha_1, \lambda) =$$

$$= L(x_{12}, x_2; \alpha_1, \lambda)\, L(x_2, x; \alpha_2, \lambda) -$$

– zero curvature representation.

Method

<u>Step 1</u>. Realize the quad-graph D as a polyhedric surface in $\mathbb{Z}^d$, where $d = \#(\text{different slopes } \alpha_1, \dots, \alpha_d)$



$$D \subset \mathbb{C} \overset{P}{\rightsquigarrow} \Omega_D \subset \mathbb{Z}^d$$

- for $x_0 \in V(D)$, set $P(x_0) = 0 \in \mathbb{Z}^d$
- if $(x,y) \in E(D)$, $y - x = \pm \alpha_i$, set $P(y) - P(x) = \pm e_i$.

<u>Step 2</u>. Rhombic $\equiv$ Integrable $\equiv$ multiD consistent.

Extend a discrete holomorphic w on $D \sim \Omega_D$ to the whole of $\mathbb{Z}^d$ by

$$\boxed{\frac{w(n+e_i+e_j) - w(n)}{w(n+e_j) - w(n+e_i)} = \frac{\alpha_i + \alpha_j}{\alpha_j - \alpha_i}}$$

Geometrically: use the flips

to fill out $\mathbb{Z}^d$, starting from $\Omega_D \subset \mathbb{Z}^d$.

Step 3. Read off the values of the extended function w on the coordinate axes: $w(me_i)$ for $m \in \mathbb{Z}$, $i=1,\dots,d$.
Note: Ω_D does not have to possess many intersection points with axes!
Thus, $\{w(me_i)\}$ is the new information about w, which we dig out of $w|_{\Omega_D}$!

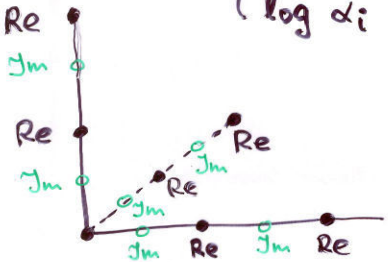
In particular:

discrete exponential function

$$e(me_i,\lambda)=\left(\frac{\lambda+\alpha_i}{\lambda-\alpha_i}\right)^m \Rightarrow e(n,\lambda)=\prod_{i=1}^{d}\left(\frac{\lambda+\alpha_i}{\lambda-\alpha_i}\right)^{n_i}$$

discrete logarithmic function

$$\ell(me_i)=\begin{cases}2\left(1+\frac{1}{3}+\frac{1}{5}+\dots+\frac{1}{2n-1}\right), & m=2n\\ \log\alpha_i, & m=2n+1\end{cases}$$



Real at •
Imaginary at ∘

Step 4. Formula for the spectral function:

$$\boxed{\rho(\lambda)\Big|_{U_i} = \frac{1}{2\lambda}\left[w(e_i) + \sum_{m=1}^{\infty}\left(\frac{\lambda-\alpha_i}{\lambda+\alpha_i}\right)^m \times \left(w((m+1)e_i) - w((m-1)e_i)\right)\right]}$$

In particular, for the discrete logarithmic function

$$\rho(\lambda)\Big|_{U_i} = \log\lambda/(2\lambda).$$

This allows us to deform the integration path from $\Gamma_1 \sqcup \ldots \sqcup \Gamma_d$ to a simple curve which has all $\pm\alpha_i$ in its interior, to use the integral representation for the asymptotic analysis, and finally to prove:

Th. The „real part" of the discrete logarithmic function (its restriction to „black" points $n=(n_1,\ldots,n_d)$ with $n_1+\ldots+n_d$ even) coincides with the discrete Green's function.

Isomonodromic property

<u>Th.</u> Discrete logarithmic function is <u>isomonodromic</u>, i.e., if $\Psi(\cdot,\lambda):\mathbb{Z}^d\to GL_2(\mathbb{C})[\lambda]$ is defined by

$$\Psi(n+e_k,\lambda)=L_k(n,\lambda)\,\Psi(n,\lambda)$$

with

$$L_k(n,\lambda)=\begin{pmatrix}\lambda+\alpha_k & -2\alpha_k\big(\ell(n+e_k)+\ell(n)\big)\\ 0 & \lambda-\alpha_k\end{pmatrix},$$

then, for a suitable choice of $\Psi(0,\lambda)$, the matrices

$$A(n,\lambda)=\frac{d\Psi(n,\lambda)}{d\lambda}\big(\Psi(n,\lambda)\big)^{-1}$$

are meromorphic in λ, with poles whose positions and orders do not depend on $n\in\mathbb{Z}^d$!

Actually, if $A(0,\lambda)=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, then

$$A(n,\lambda)=\frac{A^{(0)}(n)}{\lambda}+\sum_{i=1}^{d}\left(\frac{B^{(i)}(n)}{\lambda+\alpha_i}+\frac{C^{(i)}(n)}{\lambda-\alpha_i}\right)$$

with

$$A^{(0)}(n)=\begin{pmatrix}0&(-1)^{n_1+\dots+n_d}\\0&0\end{pmatrix}$$

$$B^{(i)}(n)=n_i\begin{pmatrix}1&-\ell(n)-\ell(n-e_i)\\0&0\end{pmatrix}$$

$$C^{(i)}(n)=n_i\begin{pmatrix}0&\ell(n)+\ell(n+e_i)\\0&1\end{pmatrix}$$

Moreover, $\forall n \in \mathbb{Z}^d$ there holds

$$\boxed{\sum_{i=1}^{d} n_i \left(\ell(n+e_i) - \ell(n-e_i) \right) = 1-(-1)^{n_1+\dots+n_d}}$$

(isomonodromic constraint, found by [Nijhoff, Ramani, Grammaticos, Ohta '2001]

<u>Sketch of the proof.</u>

Values on the axes,

$$\ell(ne_i) = \begin{cases} 2\left(1+\frac{1}{3}+\dots+\frac{1}{2m-1}\right), & n=2m \\ \log \alpha_i & , n=2m+1 \end{cases}$$

satisfy the recurrent relation

$$\boxed{n\left(f_{n+1}-f_{n-1}\right) = 1-(-1)^n} \qquad (*)$$

Matrices $A(n,\lambda)$ are defined by

$$A(n+e_k,\lambda) = \frac{dL_k(n,\lambda)}{d\lambda} L_k^{-1}(n,\lambda) + L_k(n,\lambda) A(n,\lambda) L_k^{-1}(n,\lambda)$$

One checks by induction that on the axes (*) is equivalent to $A(ne_i)$ having only $\lambda=0$ and $\lambda=\pm\alpha_i$ as <u>simple</u> poles:

$$A(ne_i,\lambda) = \frac{A^{(0)}(ne_i)}{\lambda} + \frac{B^{(i)}(ne_i)}{\lambda+\alpha_i} + \frac{C^{(i)}(ne_i)}{\lambda-\alpha_i}$$

Extend the solution from coordinate axes by dCR (d-dim consistency!) Then there cannot appear new poles, and the order of old ones cannot increase:

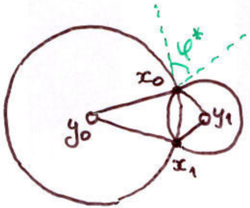
Indeed,

$$A(n+e_i+e_j,\lambda) =$$

$$= \left(\frac{dL_j(n+e_i,\lambda)}{d\lambda} + L_j(n+e_i,\lambda) A(n+e_i,\lambda) \right) L_j^{-1}(n+e_i,\lambda)$$

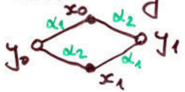
$\Rightarrow$ all poles remain simple, with the possible exception of $\lambda = \pm \alpha_j$, whose orders might increase by 1. But now $i \leftrightarrow j$: all poles remain simple! $\blacksquare$

Discrete holomorphic functions as linearization of circle patterns



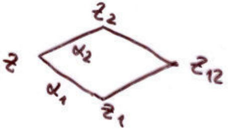
$$q(x_0, y_0, x_1, y_1) = e^{2i\varphi^*}$$

Th. Prescribing cross-ratios on all quadrilaterals leads to an integrable system on a quad-graph, iff these cross-ratios come from a rhombic embedding of this quad-graph :



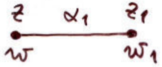
$$q(x_0, y_0, x_1, x_1) = \frac{\alpha_1^2}{\alpha_2^2}$$

In this integrable case,



$$\boxed{q(z, z_1, z_{12}, z_2) = \frac{z-z_1}{z_1-z_{12}} \, \frac{z_{12}-z_2}{z_2-z} = \frac{\alpha_1^2}{\alpha_2^2},}$$ - cross-ratio eq.

one can perform a <u>Miura</u> transformation to the <u>Hirota</u> equation:



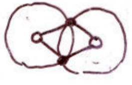
$$\boxed{z - z_1 = \alpha_1 w w_1}$$ - Miura

$$\boxed{\alpha_1 w w_1 + \alpha_2 w_1 w_{12} - \alpha_1 w_2 w_{12} - \alpha_2 w w_2 = 0}$$ - Hirota eq.

Solutions of cross-ratio eq. corresponding to circle patterns, are conveniently characterized by the Hirota variables:

$w(\circ) \in \mathbb{R}_+$ - radii

$w(\bullet) \in S^1$ - rotation relative to the rhombic realization.



The rhombic realization itself, with the edges α_i, corresponds to the trivial solution

$$w \equiv 1$$

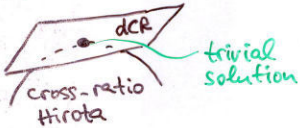
of the Hirota equation.

Let $z_\varepsilon, w_\varepsilon$ be one-parameter families of solutions of the cross-ratio, resp. Hirota systems, through the trivial solutions z_0, w_0.

<u>Th.</u> $g := \frac{dz_\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0}$ and $f := \left(w_\varepsilon^{-1} \frac{dw_\varepsilon}{d\varepsilon} \right)\Big|_{\varepsilon=0}$

Both solve the discrete Cauchy-Riemann.

So:



{Discrete holomorphic functions} = tangent space to solutions of cross-ratio and Hirota

{Discrete holomorphic functions with $f(\circ) \in \mathbb{R}$, $f(\bullet) \in \Im m$} = = tangent space to circle patterns in Hirota variables.

<u>Th.</u> Discrete logarithmic function is the tangent vector to the curve consisting of the circle patterns $z^{1+\varepsilon}$.