Moduli of stable parabolic connections, Riemann-Hilbert correspondences and Geometry of Painlevé equations

Masa-Hiko SAITO (Kobe University)

Based on Joint Works with Michi-aki Inaba and Katsunori Iwasaki (Kyushu Univ.)

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Conference on Riemann-Hilbert Problems, Integrability and Asymptotics
The purposes
- To understand the Painlevé property of equations of Painlevé type (Painlevé equations or Garnier equations, etc) from the viewpoint of the isomonodromic or isostokes deformations of linear diff. equations.
- To understand many nice properties of equations of Painlevé type like
  1. Spaces of initial conditions due to Okamoto (Sakai’s work or Saito–Takebe–Terajima’s work on Okamoto–Painlevé pair (S, Y)).
  2. Symmetries of equations = Bäklund transformations
  3. Non-autonomous Hamiltonian structures
  4. Special solutions – Riccati type solutions, Rational solutions, etc
  5. $\tau$-functions
— To understand the WKB analysis through the compactification of the moduli space of the stable parbolic connections. (Full story will be in future).
Today’s talk

- Restricted to the case of regular singularities

Papers

- **Translations of the terminology**

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<th>Geometry</th>
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<td>$C$: a compact R. surface of genus $g$</td>
<td>$C$: a nonsing. proj. curve of genus $g$</td>
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<td>$t = (t_1, \ldots, t_n)$; $n$-distinct pts on $C$</td>
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<tr>
<td>$\frac{dx}{dz} = \sum_{i=1}^{n} \frac{A_i(z)}{z-t_i} x$</td>
<td>$\nabla : E \rightarrow E \otimes \Omega^1_C(D(t))$</td>
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<td>Linear D.E. on $C$ with at most regular sing. at $t$.</td>
<td>A connection on vect. bdl $E$ of rank $r$ on $C$ with at most $1^{st}$ order poles at $t$.</td>
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<td>$\lambda_j^{(i)}$: Eigenvalues of $A_i(t_i)$</td>
<td>$\lambda_j^{(i)}$: Eigenvalues of $\text{res}<em>{t_i}(\nabla) \in \text{End}(E</em>{</td>
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<td>$T = M_{g,n} = {(C, t)}$</td>
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<td>Moduli of $n$-pointed curves of genus $g$</td>
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<td>Space of initial conditions</td>
<td>Moduli space of stable parabolic connections $M^\alpha(C, t)_\lambda$</td>
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<td>$S \rightarrow T \times \Lambda^r_n$</td>
<td>$\mathcal{M} \rightarrow T \times \Lambda^r_n$</td>
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Stable Parabolic connections

Setting

Fix the following data

\[(C, t, (L, \nabla_L), (\lambda_j^{(i)}))\] (1)

which consists of

- \(C\) : a complex smooth projective curve of genus \(g\),
- \(t = (t_1, \cdots, t_n)\): a set of \(n\)-distinct points on \(C\).
  ( Put \(D(t) = t_1 + \cdots + t_n\)).
- \((L, \nabla_L)\): a line bundle on \(C\) with a logarithmic connection

\[
\nabla_L : L \longrightarrow L \otimes \Omega^1_C(D_t).
\]

- \(\lambda = (\lambda_j^{(i)})_{1 \leq i \leq n, 0 \leq j \leq r-1} \in \mathbb{C}^{nr}\) such that \(\sum_{j=0}^{r-1} \lambda_j^{(i)} = \text{res}_{t_i}(\nabla_L)\).
Moduli space of stable parabolic connections

We can consider the moduli space of stable parabolic connections on $C$ with logarithmic singularities at $D(t)$:

\[(2) \quad \mathcal{M}^\alpha(C, t, L)_\lambda = \{(E, \nabla_E, \{l_{ij}^{(i)}\}_{1 \leq i \leq n, 0 \leq j \leq r-1, \Psi})/ \simeq \}

- $E$ : a vector bundle of rank $r$ on $C$
- $\nabla : E \rightarrow E \otimes \Omega_C(D(t))$ : a logarithmic connection
- $\Psi : \wedge^r E \overset{\sim}{\longrightarrow} L$ : a horizontal isomorphism (Fixing the determinant)
- $E|_{t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_r = 0$: a filtration of the fiber at $t_i$ such that $\dim \left( \frac{l_j^{(i)}}{l_{j+1}^{(i)}} \right) = 1$ and
  \[
  \left( \text{res}_{t_i}(\nabla) - \lambda_j^{(i)} Id \right) \left( l_j^{(i)} \right) \subset l_{j+1}^{(i)}
  \]
Moduli space of $SL_r$-rep. of the fundamental group

Take the categorical quotient of affine variety

(3)

$$\text{Rep}(C, t, r) = \{ \rho : \pi_1(C \setminus D(t)) \rightarrow SL_r(C) \} / / Ad(SL_r(C))$$

$(\rho_1, \rho_2 \in \text{Hom}(\pi_1(C \setminus D_t), SL_r(C))$ are Jordan equivalent iff $\text{sem}(\rho_1) \simeq \text{sem}(\rho_2)$).

Fix:

$$\mathbf{a} = \begin{pmatrix} a^{(i)}_{j} \\ \end{pmatrix}_{1 \leq i \leq n, 1 \leq j \leq r-1} \in \mathcal{A}_{r,n} = \mathbb{C}^{n(r-1)}$$

Then we define another moduli space of $SL_r$-representations with fixed characteristic polynomial of monodromies around $t_i$:

$$\text{Rep}(C, t, r)_\mathbf{a} = \left\{ [\rho] \in \text{Rep}(C, t, r), \det(sI_r - \rho(\gamma_i)) = \chi_{\mathbf{a}(i)}(s) \right\}$$

where

$$\chi_{\mathbf{a}(i)}(s) = s^r + a^{(i)}_{r-1}s^{r-1} + \cdots + a^{(i)}_1s + (-1)^r.$$
Riemann-Hilbert correspondence

Assume that $r \geq 2$, $n \geq 1$ and $nr - 2r - 2 > 0$ when $g = 0$, $n \geq 2$. (Moreover the weight $\alpha$ is generic). Then the Riemann-Hilbert correspondence

\[(4) \quad \text{RH}_{(C, t, \lambda)} : \mathcal{M}^\alpha(C, t, L)_\lambda \rightarrow \text{Rep}(C, t, r)_a\]

can be defined by

\[
\begin{align*}
(E, \nabla_E, \{t^{(i)}_j\}, \Psi) &\mapsto \ker(\nabla^{an}_{|C\setminus D_t})
\end{align*}
\]

where

\[
\chi_{a(i)}(s) = \prod_{j=0}^{r-1} (s - \exp(-2\pi \sqrt{-1}\lambda^{(i)}_j))
\]

Note that

\[
\dim \mathcal{M}^\alpha(C, t, L)_\lambda = (r - 1)(2(r + 1)(g - 1) + rn)
\]
Fundamental Results

Theorem 1. (Inaba-Iwasaki-Saito \( r = 2, g = 0, n \geq 4 \), Inaba (general case)) Under the notation as above, we have the following.

1. The moduli space \( \mathcal{M}^\alpha(C, t, L)_\lambda \) is a nonsingular algebraic manifold with a natural symplectic structure.

2. The moduli space \( \mathcal{M}^\alpha(C, t, L)_\lambda \) has a natural compactification \( \overline{\mathcal{M}^\alpha(C, t, L)}_\lambda \) which is the moduli space of the \( \phi \)-stable parabolic connections.
Theorem 2. (Inaba-Iwasaki-Saito ($r = 2, g = 0, n \geq 4$), Inaba (general case)): Under the conditions above, the Riemann-Hilbert correspondence

$$\text{RH}_{C,t,\lambda} : \mathcal{M}^\alpha(C, t, L)_\lambda \rightarrow \text{Rep}(C, t, r)_a$$

is a proper surjective bimeromorphic map. Hence the Riemann-Hilbert correspondence gives an (analytic) resolution of singularities. Moreover $\text{RH}_{C,t,\lambda}$ preserves the symplectic structures on $\text{Rep}(C, t, r)_a$ and $\mathcal{M}^\alpha(C, t, L)_\lambda$.

Remark 1.

- $\text{Rep}(C, t, r)_a$ is an affine scheme which may have singularities for special $a$.
- In the case of $g = 0$, we can show that $d\omega = 0$.
- Moreover, we expect that $d\omega = 0$ in general.
Varying time \((C, t)\) and parameter \(\lambda\), a

Consider the open set of the moduli space of \(n\)-pointed curves of genus \(g\)

\[
M^0_{g,n} = \{(C, t) = (C, t_1, \cdots, t_n), t_i \neq t_j, i \neq j\}
\]

and the universal curve \(\pi : C \longrightarrow M^0_{g,n}\). Fixing a relative line bundle \(L\) for \(\pi\) with logarithmic connection \(\nabla^L\) we can obtain the family of moduli spaces over \(M^0_{g,n} \times \Lambda(L)\)

\[
\begin{array}{c}
\mathcal{M}^\alpha_{g,n}(L) \\
\downarrow \quad \pi_n \\
M^0_{g,n} \times \Lambda(L)
\end{array}
\]

such that

\[
\pi_n^{-1}((C, t, L, \lambda)) = \mathcal{M}^\alpha(C, t, L)_{\lambda}
\]
We can also construct the fiber space
\[ \text{Rep}_{g}^{r,n} \]
\[ \downarrow \phi_{g}^{r,n} \]
(7)
\[ M_{g,n}^{0} \times A_{r,n} \]
such that
\[ (\phi_{g}^{r,n})^{-1}((C, t, a)) = \text{Rep}(C, t, SL_{r})a. \]
Riemann-Hilbert corr. in family

We can obtain the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{M}^\alpha(L) & \xrightarrow{\text{RH}_n} & \mathcal{R}ep^r_g \\
\downarrow\pi_n & & \downarrow\phi^r_g \\
M^o_{g,n} \times \Lambda(L) & \xrightarrow{(1 \times \mu_{r,n})} & M^o_{g,n} \times \mathcal{A}_r,n
\end{array}
\]

(8)

where \( \mu_{r,n} \) can be given by the relations

\[
\chi a(s) = \prod_{j=0}^{r-1} (s - \exp(-2\pi \sqrt{-1} \lambda_j^{(i)}))
\]

that is, \( a_k^{(i)} \) are \((\pm 1) \times k^{th}\) fundamental symmetric functions of \( \exp(-2\pi \sqrt{-1} \lambda_j^{(i)}) \).
Geometric Isomonodromic Deform. of L.D.E.
The case of generic exponents $\lambda$

Fix a generic $\lambda \in \Lambda(L)$ and set $a = \mu_{r,n}(\lambda)$ so that

$$\text{RH}_{C,t,\lambda} : \mathcal{M}^{\alpha}(C, t, L)_{\lambda} \xrightarrow{\sim} \text{Rep}(C, t, r)_a$$

is an analytic isomorphism for any $(C, t) \in M_{g,n}^0$.

- Algebraic structure of $\text{Rep}(C, t, r)_a$
  - does not change under variation of $(C, t)$, that is,

$$\text{Rep}(C, t, r)_a \simeq \text{Rep}(C_0, t_0, r)_a.$$
Taking the universal covering map $\widetilde{M}_{g,n}^o \to M_{g,n}^o$, and pulling back we obtain the diagram:

$$
\begin{align*}
\mathcal{M}_{g,n}^{\alpha}(L)_{\lambda} \xrightarrow{RH_{n,\lambda}} \left( \mathcal{R}ep_{g}^{r,n} \right)_{a} & \simeq \mathcal{R}ep(C_0, t_0, r)_{a} \times \widetilde{M}_{g,n}^o \\
\left( \tilde{\pi}_{n} \right)_{\lambda} & \downarrow \quad \quad \downarrow \tilde{r}_{n}^{r,n} \phi_{g,a} \\
\widetilde{M}_{g,n}^o \times \{ \lambda \} \xrightarrow{(1\times \mu_{r,n})} \widetilde{M}_{g,n}^o \times a.
\end{align*}
$$

Since $\tilde{r}_{n}^{r,n}$ is isomorphic to product family, it has a unique constant section $s_x$ passing through a point $x \in \mathcal{R}ep(C_0, t_0, r)_{a} \times \{ t_0 \}$.

Pulling back the section $\{ s_x \}_{x \in \mathcal{R}ep(C_0, t_0, r)_{a} \times \{ t_0 \}}$ via $RH_{\lambda}$, we obtain the set of analytic sections of $\left( \tilde{\pi}_{n} \right)_{\lambda} : \mathcal{M}_{g,n}^{\alpha}(L)_{\lambda} \to \widetilde{M}_{g,n}^o \times \{ \lambda \}$

$$
\{ \tilde{s}_x \}_{x \in \mathcal{R}ep(C_0, t_0, r)_{a} \times \{ t_0 \}}.
$$
The family of sections \( \{ \tilde{s}_x \}_x \) gives the splitting homomorphism
\[
\tilde{v}_\lambda : (\tilde{\pi}_n)^*(T_{\overline{M}_{g,n}}^o \times \{ \lambda \}) \rightarrow T_{\overline{M}_{g,n}(L)_\lambda}^\alpha
\]
for the natural homomorphism \( T_{\overline{M}_{g,n}(L)_\lambda}^\alpha \rightarrow (\tilde{\pi}_n)^*(T_{\overline{M}_{g,n}}^o \times \{ \lambda \}) \).

Then the subbundle
\[
(9) \quad \mathcal{IF}_{g,n,\lambda} = \tilde{v}_\lambda((\tilde{\pi}_n)^*(T_{\overline{M}_{g,n}}^o \times \{ \lambda \})) \subset T_{\overline{M}_{g,n}(L)_\lambda}^\alpha.
\]

Take any local generators of the tangent sheaf of \( T_{\overline{M}_{g,n}}^o \)
\[
\langle \frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_N} \rangle.
\]
where \( N = 3g - 3 + n = \dim \overline{M}_{g,n}^o \). Then setting \( v_i(\lambda) := v_\lambda(\frac{\partial}{\partial q_i}) \),
we obtain the integrable differential system on \( \overline{M}_{g,n}(L)_\lambda \)
\[
\mathcal{IF}_{g,n,\lambda} \cong \langle v_1(\lambda), \ldots, v_N(\lambda) \rangle.
\]
(locally).
Case of special exponents $\lambda$

- When the set of exponents $\lambda$ is special, the R.H. corr.

$$\text{RH}_{n,\lambda} : \mathcal{M}^{\alpha, \lambda}_{g, n}(L) \to \left( \mathcal{R}ep^{r, n}_g \right)_a$$

contracts some subvarieties to the singular locus on $\left( \mathcal{R}ep^{r, n}_g \right)_a$.

- However, by Hartogs’ theorem, we can extend the isomonodromic foliation $\mathcal{IF}_{g, n, \lambda}$ to the total space $\mathcal{M}^{\alpha}_{g, n}(L)_{\lambda}$. 
Painlevé Property of Isomonodromic Flows

Theorem 3. The isomonodromic flows $\mathcal{IF}_\lambda$ satisfies the Painlevé property for all exponents $\lambda$.

Hamiltonian structure of Isomonodromic Flows

Theorem 4. The isomonodromic flows $\mathcal{IF}_\lambda$ can be written in a Hamiltonian system locally.
• In the case of generic $\lambda$, the differential system on $\hat{M}_{g,n}(L)^\alpha_\lambda$

$$\mathcal{IF}_{g,n,r} := \langle v_1(\lambda), \ldots, v_N(\lambda) \rangle.$$ 

has clearly solution manifolds or integrable manifolds = the images of $\hat{M}_{g,n}^o$ by $\{\tilde{s}_x\}_x$. By construction,

These integrable submanifolds are isomonodromic flow of connections.

• Even in the case of special $\lambda$, the properness of $\text{RH}_{\lambda,n}$ implies the theorem.

• $\mathcal{IF}_{(0,4,2)}$ is equivalent to a Painlevé VI equation.

• $\mathcal{IF}_{(0,n,2)}$ with $n \geq 5$ are Garnier systems.
Painlevé VI case
We will see what is happening in the case of Painlevé VI equations.
Isomonodromic flows = Painlevé or Garnier flows

\[ M_n^\alpha(t_0, \lambda, L) \quad \overset{\text{RH}_\lambda}{\approx} \quad M_n^\alpha(t, \lambda, L) \]

\[ \mathcal{R}(P_{n,t})_a \]

\[ \mathcal{R}(P_{n,t_0})_a \]

\[ t_0 \quad T_n \times \{ \lambda \} \quad t \quad T_n \times \{ a \} t \]

Figure 1. Riemann-Hilbert correspondence and isomonodromic flows for generic \( \lambda \)
Riccati flows are tangent to family of \((-2)\)-curves

\[ \tilde{T}_4 \times \{ \lambda \} \]

\[ M_4^\alpha(t_0, \lambda, L) \]

\[ \text{contraction} \]

\[ \text{Case of Painlevé VI} \]

Figure 2. Riemann-Hilbert correspondence and isomonodromic flows for special \( \lambda \)
Hamiltonian systems of Painlevé $P_{VI}$

$P_{VI}$ is equivalent to a Hamiltonian system $H_{VI}$.

$$(H_{VI}): \begin{cases} \frac{dx}{dt} = \frac{\partial H_{VI}}{\partial y}, \\ \frac{dy}{dt} = -\frac{\partial H_{VI}}{\partial x}, \end{cases}$$

Hamiltonian in suitable coordinates can be given

$$H_{VI} = H_{VI}(x, y, t, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{C}(t)[x, y, \lambda]$$

$$H_{VI}(x, y, t) = \frac{1}{t(t-1)} \left[x(x-1)(x-t)y^2 - \{2\lambda_1(x-1)(x-t) + 2\lambda_2x(x-t) + (2\lambda_3 - 1)x(x-1)\}y + \lambda(x-t)\right]$$

$$\lambda := \{(\lambda_1 + \lambda_2 + \lambda_3 - 1/2)^2 - \lambda_4^2\}.$$
Bäcklund transformations for Painlevé VI.

• $P_{VI}(\lambda)$ have non-trivial birational automorphisms which are called Bäcklund transformations. The group of all Bäcklund transformations form the affine Weyl group $W$ of type $D_4^{(1)}$.

**Proposition 1.** The group of Bäcklund transformations which can be obtained from elementary transformations of stable parabolic connections is a proper subgroup of $W(D_4^{(1)})$ whose index is finite. The invloution $s_0$ of $W(D_4^{(1)})$ is not in the group.
Theorem 5. 1. For a suitable choice of a weight \( \alpha' \), the morphism

\[
\pi_4 : \overline{M_4^{\alpha'}}(-1) \longrightarrow T_4 \times \Lambda_4
\]

is projective and smooth. Moreover for any \((t, \lambda) \in T_4 \times \Lambda_4\) the fiber \(\pi_4^{-1}(t, \lambda) := \overline{M_4^{\alpha'}}(t, \lambda, -1)\) is irreducible, hence a smooth projective surface.

2. Let \( \mathcal{D} = \overline{M_4^{\alpha'}}(-1) \setminus \overline{M_4^{\alpha}}(-1) \) be the complement of \( \overline{M_4^{\alpha}}(-1) \) in \( \overline{M_4^{\alpha'}}(-1) \).
(Note that \( \alpha = \alpha' / 2 \)). Then \( \mathcal{D} \) is a flat reduced divisor over \( T_4 \times \Lambda_4 \).

3. For each \((t, \lambda)\), set\n
\[
\overline{S}_{t, \lambda} := \pi_4^{-1}(t, \lambda) := \overline{M_4^{\alpha'}}(t, \lambda, -1).
\]

Then \( \overline{S}_{t, \lambda} \) is a smooth projective surface which can be obtained by blowing-ups at 8 points of the Hirzeburch surface \( \mathbf{F}_2 = \text{Proj}(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}) \) of degree 2. The surface has a unique effective anti-canonical divisor \(-K_{S_{t, \lambda}} = \mathcal{Y}_{t, \lambda}\) whose support is \( \mathcal{D}_{t, \lambda} \). Then the pair

(10) \( (\overline{S}_{t, \lambda}, \mathcal{Y}_{t, \lambda}) \)

is an Okamoto-Painlevé pair of type \( D_4^{(1)} \). That is, the anti-canonical divisor \( \mathcal{Y}_{t, \lambda} \) consists of 5-nodal rational curves whose configuration is same as Kodaira–Néron degenerate elliptic curves of type \( D_4^{(1)} \) (=Kodaira type \( I_0^* \)).
Moreover we have \( (\overline{M_4^{\alpha}}(-1))_{t, \lambda} = (\overline{M_4^{\alpha'}}(-1))_{t, \lambda} \setminus \mathcal{Y}_{t, \lambda} \).
Figure 3. Okamoto-Painlevé pair of type $D_{4}^{(1)}$
Proposition 2. The invariant ring \((R_3)^{Ad(SL_2(C))}\) is generated by seven elements \(x_1, x_2, x_3, a_1, a_2, a_3, a_4\) and there exist a relation
\[
f(x, a) = x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1(a)x_1 - \theta_2(a)x_2 - \theta_3(a)x_3 + \theta_4(a),
\]
where we set
\[
\begin{align*}
\theta_i(a) &= a_ia_4 + a_ja_k, \\
\theta_4(a) &= a_1a_2a_3a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4.
\end{align*}
\]
Therefore we have an isomorphism
\[
(R_3)^{Ad(SL_2(C))} \cong \mathbb{C}[x_1, x_2, x_3, a_1, a_2, a_3, a_4]/(f(z, a)).
\]
Hence
\[
\text{Rep}(\mathbb{P}^1, (t_1, t_2, t_3, t_4), 2)\mathfrak{a} = \text{Spec } (R_3)^{Ad(SL_2(C))}
\]
is isomorphic to an affine cubic.

(12) \[ a_i = 2\cos 2\pi \lambda_i \] for \(1 \leq i \leq 4.\)
The family of affine cubic surfaces

\[ a_1 = 2 \]

\[ \Delta = 0 \]

\[ A_1 \text{-singularity} \]

\[ R(\mathcal{P}_{4,t}) \]

\[ a_i = 2 \]

\[ A_4 \cong \mathbb{C}^4 \]
\[ M_\alpha^n (L) \xrightarrow{\text{RH}_n} \mathcal{R}_n \]

\[ \pi_n \downarrow \quad \downarrow \phi_n \]

\[ T'_n \times \Lambda_n \xrightarrow{(1 \times \mu_n)} T'_n \times \mathcal{A}_n. \]

Here, we have \( 1 \times \mu_n \; (1 \times \mu_n)(t, \lambda) = (t, a) \)

\[ a_i = 2 \cos 2\pi \lambda_i \]

\text{for} \; 1 \leq i \leq n.