RHPIA-2005

Moduli of stable parabolic connections, Riemann-Hilbert correspondences and Geometry of Painlevé equations

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Based on Joint Works with
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Conference on Riemann-Hilbert Problems, Integrability and Asymptotics

- **The purposes**
  - To understand the Painlevé property of equations of Painlevé type (Painlevé equations or Garnier equations, etc) from the view-point of the isomonodromic or isostokes deformations of linear diff. equations.
  - To understand many nice properties of equations of Painlevé type like
    1. Spaces of initial conditions due to Okamoto (Sakai's work or Saito–Takebe –Terajima's work on Okamoto–Painlevé pair (S, Y)).
    2. Symmetries of equations $=$ Bäklund transformations
    3. Non-autonomous Hamiltonian structures
    4. Special solutions – Riccati type solutions, Rational solutions, etc
    5. τ-functions

– To understand the WKB analysis through the compactification of the moduli space of the stable parbolic connections. (Full story will be in future).

Today's talk

- Restricted to the case of regular singularities

Papers

- M. Inaba, K. Iwasaki and M.-H. Saito, *Bäcklund transformations of the sixth Painlevé equation in terms of Riemann-Hilbert Correspondence*, Internat. Math. Res. Notices **2004:1** (2004), 1–30.
- M. Inaba, K. Iwasaki and M.-H. Saito, *Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painlevé equation of type VI. Part I*, math.AG/0309342 (2003); *Part II*, in preparation.
- M. Inaba, K. Iwasaki and M.-H. Saito, Dynamics of the Sixth Painlevé Equation, to appear in Angers proceedings, math.AG/0501007

• Translations of the terminology

Analysis	Geometry
C: a compact R. surface of genus g	C: a nonsing. proj. curve of genus g
$\mathbf{t} = (t_1, \cdots, t_n)$; n-distinct pts on C	$\mathbf{t} = (t_1, \cdots, t_n)$; n-distinct pts on C
$\frac{d\mathbf{x}}{dz} = \sum_{i=1}^n \frac{A_i(z)}{z-t_i}\mathbf{x}$ Linear D.E. on C with at most regular sing. at $\mathbf{t}$.	$\nabla : E \longrightarrow E \otimes \Omega^1_C(D(\mathbf{t}))$ A connection on vect. bdl E of rank r on C with at most 1^{st} order poles at $\mathbf{t}$.
$\lambda_j^{(i)}$:Eigenvalues of $A_i(t_i)$	$\lambda_j^{(i)}$: Eigenvalues of $\mathrm{res}_{t_i}(\nabla) \in \mathrm{End}(E_{\|t_i})$
Time varaiables $(s_1, \ldots, s_{3g-3}, t_1, \ldots, t_n)$	$T = \mathcal{M}_{g,n} = \{(C, \mathbf{t})\}$ Moduli of n-pointed curves of genus g
Space of initial conditions $S_{(C,\mathbf{t},\boldsymbol{\lambda})}$	Moduli space of stable parabolic connections $\mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t})_{\boldsymbol{\lambda}}$
Phase space $\mathcal{S} \longrightarrow T \times \Lambda_n^r$	Family of moduli spaces $\mathcal{M} \longrightarrow T \times \Lambda_n^r$
Riemann-Hilbert correspondence	$\mathbf{RH}_{\boldsymbol{\lambda}} : \mathcal{M}^{\boldsymbol{\alpha}}_{\boldsymbol{\lambda}} \longrightarrow R_{\mathbf{a}}$
Isomonodromic deformations of L.D.E.	Pullback of local constant section
Schlessinger equation	Zero curvature equations on $\mathcal{M}$

• Translations of Properties

Analysis	Geometry
Painlevé property	Properness + Surjectivity of $\mathbf{RH}_{\boldsymbol{\lambda}} : \mathcal{M}_{\boldsymbol{\lambda}}^{\boldsymbol{\alpha}} \longrightarrow R_{\mathbf{a}}$
Symmetry (Bäklund transformation)	Elementary transformations of s.p. conn.
Simple reflections in Bäcklund transf.	Special Birational map (Flop) $\tilde{s} : \mathcal{M} \cdots \longrightarrow \mathcal{M}$ appeared in the resol. of simult. sing. of $R_{\mathbf{a}}$
Hamitonian Structures	Symplectic str. on $\mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t})_{\boldsymbol{\lambda}}$ on $R_{\mathbf{a}}^{smooth}$ and $\mathbf{RH}_{\boldsymbol{\lambda}}$ is a symmplectic map
Special solutions like Riccati solution	Singylarities of $R_{\mathbf{a}}$
Poincaré return map or non-linear monodromy of equations of Painlevé type	Natural actions of $\pi_1(\mathcal{M}_{g,n}^{\circ}, *)$ on isomonodromic flows, $\mathbf{R}_{(C_0, \mathbf{t}_0), \mathbf{a}}$ and on $\mathcal{M}^{\boldsymbol{\alpha}}((C_0, \mathbf{t}_0))_{\boldsymbol{\lambda}}$
τ-functions	Sections of the determinant line bundle on $\mathcal{M}$ which are flat on isomonod. flows

Stable Parabolic connections

Setting

Fix the following data

$$(C, \mathbf{t}, (L, \nabla_L), (\lambda_j^{(i)})) \tag{1}$$

which consists of

- C : a complex smooth projective curve of genus g,
- $\mathbf{t} = (t_1, \cdots, t_n)$: a set of n-ditinct points on C.
  (Put $D(\mathbf{t}) = t_1 + \cdots + t_n$).
- (L, ∇_L): a line bundle on C with a logarithmic connection

$$\nabla_L : L \longrightarrow L \otimes \Omega^1_C(D_{\mathbf{t}}).$$

- $\boldsymbol{\lambda} = (\lambda_j^{(i)})_{1 \leq i \leq n, 0 \leq j \leq r-1} \in \mathbf{C}^{nr}$ such that $\sum_{j=0}^{r-1} \lambda_j^{(i)} = \mathrm{res}_{t_i}(\nabla_L)$.

Moduli space of stable parabolic connections

We can consider the moduli space of stable parabolic connections on C with logarithmic singularities at $D(\mathbf{t})$:

$$(2) \qquad \mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t}, L)_{\boldsymbol{\lambda}} = \{(E, \nabla_E, \{l_j^{(i)}\}_{1 \le i \le n, 0 \le j \le r-1}, \Psi)\}/\simeq$$

- E : a vector bundle of rank r on C
- $\nabla : E \longrightarrow E \otimes \Omega_C(D(\mathbf{t}))$:a logarithmic connection
- $\Psi : \wedge^r E \xrightarrow{\simeq} L$: a horizontal isomorphism (Fixing the determinant)
- $E_{|t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_r = 0$: a filtration of the fiber at t_i such that $\dim \left(l_j^{(i)} / l_{j+1}^{(i)} \right) = 1$ and
$$\left(\mathrm{res}_{t_i}(\nabla) - \lambda_j^{(i)} Id \right) (l_j^{(i)}) \subset l_{j+1}^{(i)}$$

Moduli space of SL_r-rep. of the fundamental group

Take the categorical quotient of affine variety

(3)
$$\mathbf{Rep}(C, \mathbf{t}, r) = \{\rho : \pi_1(C \setminus D(\mathbf{t})) \longrightarrow SL_r(\mathbf{C})\}//Ad(SL_r(\mathbf{C}))$$

($\rho_1, \rho_2 \in \mathrm{Hom}(\pi_1(C \setminus D_{\mathbf{t}}), SL_r(\mathbf{C}))$ are Jordan equivalent iff $\mathrm{sem}(\rho_1) \simeq \mathrm{sem}(\rho_2)$).

Fix:
$$\mathbf{a} = \left(a_j^{(i)}\right)_{1 \leq i \leq n, 1 \leq j \leq r-1} \in \mathcal{A}_{r,n} = \mathbf{C}^{n(r-1)}$$

Then we define another moduli space of SL_r-representations with fixed characteristic polynomial of monodromies around t_i:

$$\mathbf{Rep}(C, \mathbf{t}, r)_{\mathbf{a}} = \left\{ [\rho] \in \mathbf{Rep}(C, \mathbf{t}, r), \det(sI_r - \rho(\gamma_i)) = \chi_{\mathbf{a}^{(i)}}(s) \right\}$$

where

$$\boxed{\chi_{\mathbf{a}^{(i)}}(s) = s^r + a_{r-1}^{(i)} s^{r-1} + \cdots + a_1^{(i)} s + (-1)^r.}$$

Riemann-Hilbert correspondence

Assume that $r \geq 2$, $n \geq 1$ and $nr - 2r - 2 > 0$ when $g = 0$, $n \geq 2$. (Moreover the weight $\boldsymbol{\alpha}$ is generic). Then the Riemann-Hilbert correspondence

$$\mathbf{RH}_{(C,\mathbf{t},\boldsymbol{\lambda})} : \mathcal{M}^{\boldsymbol{\alpha}}(C,\mathbf{t},L)_{\boldsymbol{\lambda}} \longrightarrow \mathbf{Rep}(C,\mathbf{t},r)_{\mathbf{a}} \tag{4}$$

can be defined by

$$\boxed{(E, \nabla_E, \{l_j^{(i)}\}, \Psi) \mapsto \ker(\nabla^{an}_{|C \setminus D_{\mathbf{t}}})}$$

where

$$\chi_{\mathbf{a}^{(i)}}(s) = \prod_{j=0}^{r-1}(s - \exp(-2\pi\sqrt{-1}\lambda_j^{(i)}))$$

Note that

$$\dim \mathcal{M}^{\boldsymbol{\alpha}}(C,\mathbf{t},L)_{\boldsymbol{\lambda}} = (r-1)(2(r+1)(g-1) + rn)$$

Fundamental Results

Theorem 1. (Inaba-Iwasaki-Saito $(r = 2, g = 0, n \geq 4)$, Inaba (general case)) Under the notation as above, we have the following.

1. The moduli space $\mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t}, L)_{\boldsymbol{\lambda}}$ is a nonsingular algebraic manifold with a natural symplectic structure.
2. The modulis space $\mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t}, L)_{\boldsymbol{\lambda}}$ has a natural compactification $\overline{\mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t}, L)_{\boldsymbol{\lambda}}}$ which is the moduli space of the ϕ-stable parabolic connections.

Theorem 2. (Inaba-Iwasaki-Saito ($r = 2, g = 0, n \geq 4$), Inaba (general case)): Under the conditions above, the Riemann-Hilbert correspondense

$$\mathbf{RH}_{C,t,\boldsymbol{\lambda}} : \mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t}, L)_{\boldsymbol{\lambda}} \longrightarrow \mathbf{Rep}(C, \mathbf{t}, r)_{\mathbf{a}} \tag{5}$$

is a proper surjective bimeromorphic map. Hence the Riemann-Hilbert correspondence gives an (analytic) resolution of singularities. Moreover $\mathbf{RH}_{C,t,\boldsymbol{\lambda}}$ preserves the symplectic structures on $\mathbf{Rep}(C, \mathbf{t}, r)_{\mathbf{a}}$ $\mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t}, L)_{\boldsymbol{\lambda}}$.

Remark 1.

- $\boxed{\text{$\mathbf{Rep}(C, \mathbf{t}, r)_{\mathbf{a}}$ is an affine scheme which may have singularities for special $\mathbf{a}$.}}$
- $\boxed{\text{In the case of $g = 0$, we can show that $d\omega = 0$. Moreover, we expect that $d\omega = 0$ in general.}}$

Varying time $(C, \mathbf{t})$ and parameter $\boldsymbol{\lambda}$, a

Consider the open set of the moduli space of n-pointed curves of genus g

$$M^o_{g,n} = \{(C, \mathbf{t}) = (C, t_1, \cdots, t_n), t_i \neq t_j, i \neq j\}$$

and the universal curve $\pi : \mathcal{C} \longrightarrow M^o_{g,n}$. Fixing a relative line bundle L for π with logarithmic connection ∇_L we can obtain the family of moduli spaces over $M^o_{g,n} \times \Lambda(L)$

$$(6) \qquad \begin{array}{c} \mathcal{M}^{\boldsymbol{\alpha}}_{g,n}(L) \\ \downarrow \pi_n \\ M^o_{g,n} \times \Lambda(L) \end{array}$$

such that $\boxed{\pi_n^{-1}((C, \mathbf{t}, L, \boldsymbol{\lambda})) = \mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t}, L)_{\boldsymbol{\lambda}}}$

We can also construct the fiber space

$$\begin{array}{ll} & \mathcal{R}\mathbf{ep}_g^{r,n} \\ (7) & \downarrow \phi_g^{r,n} \\ & M_{g,n}^o \times \mathcal{A}_{r,n} \end{array}$$

such that

$$(\phi_g^{r,n})^{-1}((C, \mathbf{t}, \mathbf{a})) = \mathbf{Rep}(C, \mathbf{t}, SL_r)_{\mathbf{a}}.$$

Riemann-Hilbert corr. in family

We can obtain the following commutative diagram:

$$
(8) \qquad \begin{array}{ccc}
\mathcal{M}^{\boldsymbol{\alpha}}(L) & \xrightarrow{\mathbf{RH}_n} & \mathcal{R}\mathbf{ep}_g^{r,n} \\
{\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \phi_g^{r,n}} \\
M^o_{g,n} \times \Lambda(L) & \xrightarrow{(1\times\mu_{r,n})} & M^o_{g,n} \times \mathcal{A}_{r,n}
\end{array}
$$

where $\mu_{r,n}$ can be given by the relations

$$\chi_{\mathbf{a}}(s) = \prod_{j=0}^{r-1}(s - \exp(-2\pi\sqrt{-1}\lambda_j^{(i)}))$$

that is, $a_k^{(i)}$ are $(\pm 1)\times$ k^{th} fundamental symmetric functions of $\exp(-2\pi\sqrt{-1}\lambda_j^{(i)})$.

Geometric Isomonodromic Deform. of L.D.E.
The case of generic exponents $\boldsymbol{\lambda}$

Fix a generic $\boldsymbol{\lambda} \in \Lambda(L)$ and set $\mathbf{a} = \mu_{r,n}(\boldsymbol{\lambda})$ so that

$$\mathbf{RH}_{C,t,\boldsymbol{\lambda}} : \mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t}, L)_{\boldsymbol{\lambda}} \xrightarrow{\simeq} \mathbf{Rep}(C, \mathbf{t}, r)_{\mathbf{a}}$$

is an analytic isomorphism for any $(C, \mathbf{t}) \in M^{o}_{g,n}$.

- Algebraic structure of $\mathbf{Rep}(C, \mathbf{t}, r)_{\mathbf{a}}$ does not change under variation of $(C, \mathbf{t})$, that is, $\mathbf{Rep}(C, \mathbf{t}, r)_{\mathbf{a}} \simeq \mathbf{Rep}(C_0, \mathbf{t}_0, r)_{\mathbf{a}}$.

- Algebraic structure of $\mathcal{M}^{\boldsymbol{\alpha}}(C, \mathbf{t}, L)_{\boldsymbol{\lambda}}$ change under variation of $(C, \mathbf{t})$.

Taking the universal covering map $\widetilde{M^o_{g,n}} \longrightarrow M^o_{g,n}$, and pulling back we obtain the diagram:

$$\begin{tikzcd}
\widetilde{\mathcal{M}^{\boldsymbol{\alpha}}_{g,n}(L)}_{\boldsymbol{\lambda}} \arrow[r, "\mathbf{RH}_{n,\boldsymbol{\lambda}}", "\simeq"'] \arrow[d, "(\tilde{\pi}_n)_{\boldsymbol{\lambda}}"'] & \left(\widetilde{\mathcal{R}ep^{r,n}_g}\right)_{\mathbf{a}} \simeq \mathbf{Rep}(C_0, \mathbf{t}_0, r)_{\mathbf{a}} \times \widetilde{M^o_{g,n}} \arrow[d, "\tilde{\phi}^{r,n}_{g,\mathbf{a}}"] \\
\widetilde{M^o_{g,n}} \times \{\boldsymbol{\lambda}\} \arrow[r, "(1\times\mu_{r,n})"] & \widetilde{M^o_{g,n}} \times \ \mathbf{a}.
\end{tikzcd}$$

Since $\tilde{\phi}^{r,n}_{g,\mathbf{a}}$ is isomorphic to product family, it has a unique constant section $s_{\mathbf{x}}$ passing through a point $\mathbf{x} \in \mathbf{Rep}(C_0, \mathbf{t}_0, r)_{\mathbf{a}} \times \{\mathbf{t}_0\}$.

Pulling back the section $\{s_{\mathbf{x}}\}_{\mathbf{x} \in \mathbf{Rep}(C_0, \mathbf{t}_0, r)_{\mathbf{a}} \times \{\mathbf{t}_0\}}$ via $\mathbf{RH}_{\boldsymbol{\lambda}}$, we obtain the set of analytic sections of $(\tilde{\pi}_n)_{\boldsymbol{\lambda}} : \widetilde{\mathcal{M}^{\boldsymbol{\alpha}}_{g,n}(L)}_{\boldsymbol{\lambda}} \to \widetilde{M^o_{g,n}} \times \{\boldsymbol{\lambda}\}$

$$\{\tilde{s_{\mathbf{x}}}\}_{\mathbf{x} \in \mathbf{Rep}(C_0, \mathbf{t}_0, r)_{\mathbf{a}} \times \{\mathbf{t}_0\}}.$$

The family of sections $\{\tilde{s}_{\mathbf{x}}\}_{\mathbf{x}}$ gives the splitting homomorphism

$$\tilde{v}_{\boldsymbol{\lambda}} : (\tilde{\pi}_n)^*_{\boldsymbol{\lambda}}(T_{\widetilde{M^o_{g,n}} \times \{\boldsymbol{\lambda}\}}) \longrightarrow T_{\widetilde{\mathcal{M}^{\boldsymbol{\alpha}}_{g,n}(L)}_{\boldsymbol{\lambda}}}$$

for the natural homomorphism $T_{\widetilde{\mathcal{M}^{\boldsymbol{\alpha}}_{g,n}(L)}_{\boldsymbol{\lambda}}} \longrightarrow (\tilde{\pi}_n)^*_{\boldsymbol{\lambda}}(T_{\widetilde{M^o_{g,n}}} \times \{\boldsymbol{\lambda}\})$.

Then the subbundle

$$\mathcal{IF}_{g,n,\boldsymbol{\lambda}} = \tilde{v}_{\boldsymbol{\lambda}}((\tilde{\pi}_n)^*_{\boldsymbol{\lambda}}(T_{\widetilde{M^o_{g,n}} \times \{\boldsymbol{\lambda}\}})) \subset T_{\widetilde{\mathcal{M}^{\boldsymbol{\alpha}}_{g,n}(L)}_{\boldsymbol{\lambda}}}. \tag{9}$$

Take any local generators of the tangent sheaf of $T_{\widetilde{M^o_{g,n}}}$

$$\langle \frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_N} \rangle.$$

where $N = 3g - 3 + n = \dim \widetilde{M^o_{g,n}}$. Then setting $v_i(\boldsymbol{\lambda}) := v_{\boldsymbol{\lambda}}(\frac{\partial}{\partial q_i})$, we obtain the integrable differential system on $\widetilde{\mathcal{M}^{\boldsymbol{\alpha}}_{g,n}(L)}_{\boldsymbol{\lambda}}$

$$\mathcal{IF}_{g,n,\boldsymbol{\lambda}} \simeq \langle v_1(\boldsymbol{\lambda}), \dots, v_N(\boldsymbol{\lambda}) \rangle.$$

(locally).

Case of special exponents $\boldsymbol{\lambda}$

- When the set of exponents $\boldsymbol{\lambda}$ is special, the R.H. corr.

$$\mathbf{RH}_{n,\boldsymbol{\lambda}} : \widetilde{\mathcal{M}^{\boldsymbol{\alpha}}_{g,n}(L)}_{\boldsymbol{\lambda}} \longrightarrow \left(\widetilde{\mathcal{R}ep^{r,n}_g}\right)_{\mathbf{a}}$$

contracts some subvatieties to the singular locus on $\left(\widetilde{\mathcal{R}ep^{r,n}_g}\right)_{\mathbf{a}}$

- However, by Hartogs' theorem, we can extend the isomonodromic foliation $\mathcal{IF}_{g,n,\boldsymbol{\lambda}}$ to the total space $\widetilde{\mathcal{M}^{\boldsymbol{\alpha}}_{g,n}(L)}_{\boldsymbol{\lambda}}$.

Painlevé Property of Isomonodromic Flows

Theorem 3. The isomonodromic flows $\mathcal{IF}_{\boldsymbol{\lambda}}$ satisfies the Painlevé property for all exponents $\boldsymbol{\lambda}$.

Hamiltonian strucure of Isomonodromic Flows

Theorem 4. The isomonodromic flows $\mathcal{IF}_{\boldsymbol{\lambda}}$ can be written in a Hamiltonian system locally.

- In the case of generic $\boldsymbol{\lambda}$, the differential system on $\widetilde{\mathcal{M}^{\boldsymbol{\alpha}}_{g,n}(L)}_{\boldsymbol{\lambda}}$

$$\mathcal{IF}_{g,n,r} := \langle v_1(\boldsymbol{\lambda}), \ldots, v_N(\boldsymbol{\lambda})\rangle.$$

has cleary solution manifolds or integrable manifolds $=$ the images of $\widetilde{M^o_{g,n}}$ by $\{\tilde{s}_{\mathbf{x}}\}_{\mathbf{x}}$. By construction,

These integrable submanifolds are isomonodromic flow of connections.

- Even in the case of special $\boldsymbol{\lambda}$, the properness of $\mathbf{RH}_{\boldsymbol{\lambda},n}$ implies the theorem.
- $\mathcal{IF}_{(0,4,2)}$ is equivalent to a Painlevé VI equation.
- $\mathcal{IF}_{(0,n,2)}$ with $n \geq 5$ are Garnier systems.

Painlevé VI case

We will see what is happening in the case of Painlevé VI equations.

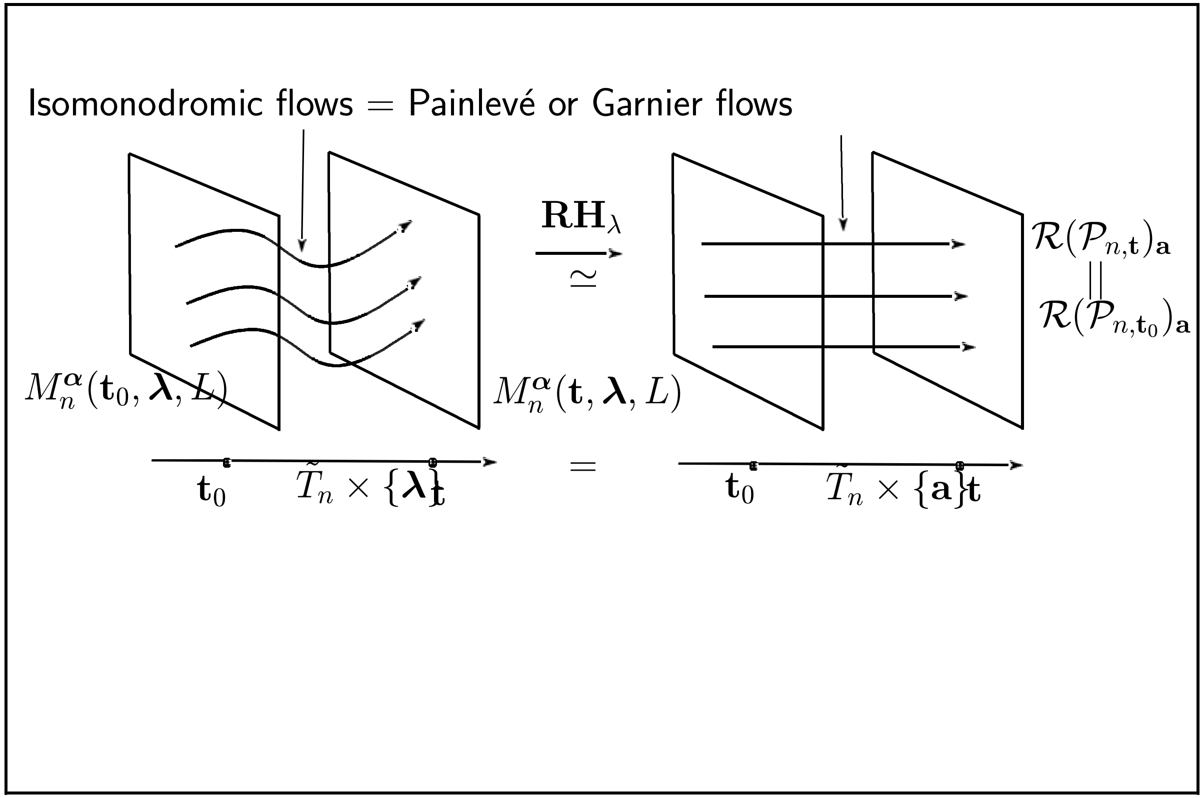


Figure 1. Riemann-Hilbert correspondence and isomonodromic flows for generic $\boldsymbol{\lambda}$

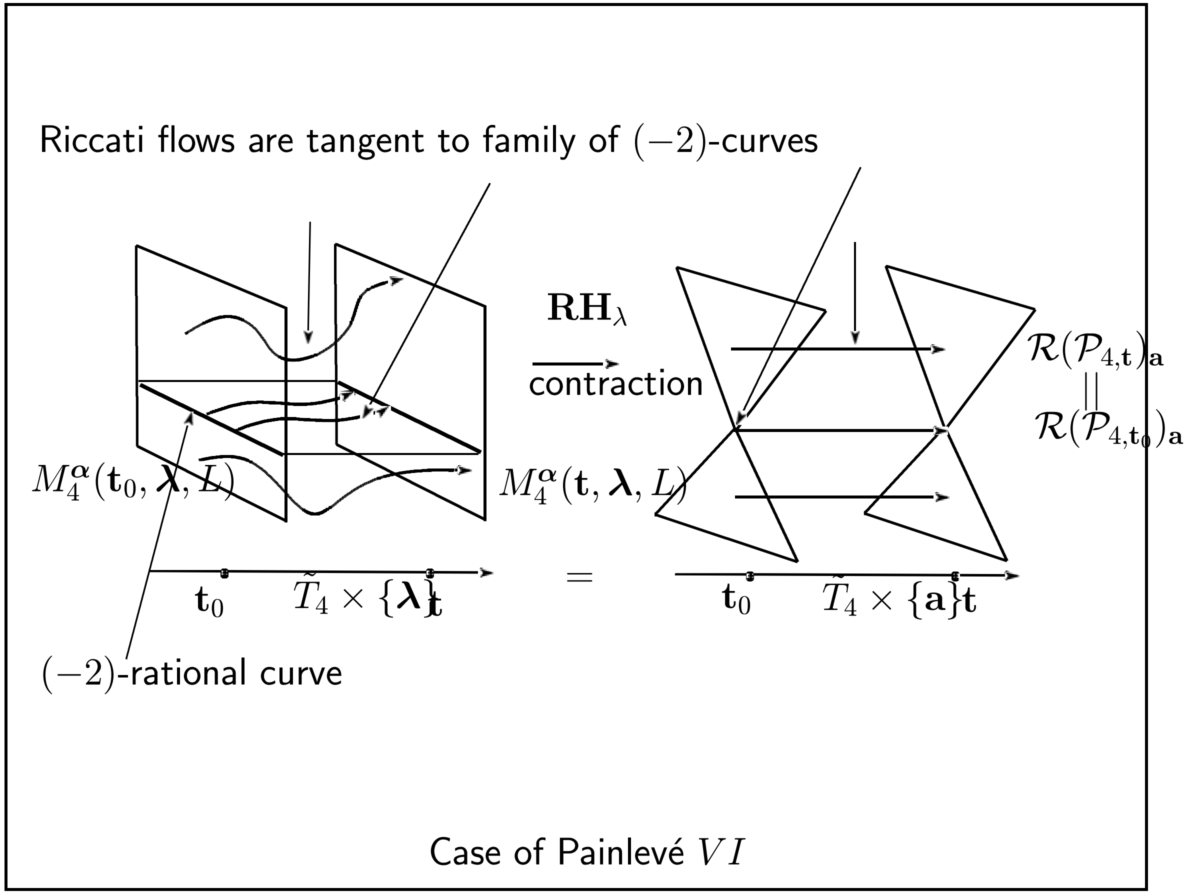


FIGURE 2. Riemann-Hilbert correspondence and isomonodromic flows for special $\boldsymbol{\lambda}$

Hamiltonian systems of Painlevé P_{VI}

P_{VI} is equivalent to a Hamiltonian system H_{VI}.

$$(H_{VI}) : \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H_{VI}}{\partial y}, \\ \dfrac{dy}{dt} = -\dfrac{\partial H_{VI}}{\partial x}, \end{cases}$$

Hamiltionian in suitable coordinates can be given

$$H_{VI} = H_{VI}(x, y, t, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbf{C}(t)[x, y, \lambda_i]$$

$$\begin{aligned} H_{VI}(x,y,t) \;&=\; \frac{1}{t(t-1)} \left[x(x-1)(x-t)y^2 - \{2\lambda_1(x-1)(x-t) \right. \\ &\left. \quad +2\lambda_2 x(x-t) + (2\lambda_3 - 1)x(x-1)\} \, y + \lambda(x-t) \right] \end{aligned}$$

$$\left(\lambda := \left\{(\lambda_1 + \lambda_2 + \lambda_3 - 1/2)^2 - \lambda_4^2\right\}\right).$$

Bäcklund transformations for Painlevé VI.

- $P_{VI}(\boldsymbol{\lambda})$ have non-trivial birational automorphisms which are called **Bäcklund transformations**. The group of all **Bäcklund transformations** form the affine Weyl group W of type $D_4^{(1)}$.

Proposition 1. *The group of Bäcklund transformations which can be obtained from elementary transformations of stable parabolic connections is a proper subgroup of* $W(D_4^{(1)})$ *whose index is finite. The invloution* s_0 *of* $W(D_4^{(1)})$ *is not in the group.*

Theorem 5. 1. *For a suitable choice of a weight $\boldsymbol{\alpha}'$, the morphism*

$$\overline{\pi}_4 : \overline{M_4^{\boldsymbol{\alpha}'}}(-1) \longrightarrow T_4 \times \Lambda_4$$

is $\underline{\text{projective and smooth}}$. Moreover for any $(\mathbf{t}, \boldsymbol{\lambda}) \in T_4 \times \Lambda_4$ the fiber $\overline{\pi}_4^{-1}(\mathbf{t}, \boldsymbol{\lambda}) := \overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$ is irreducible, hence a smooth projective surface.

2. *Let $\mathcal{D} = \overline{M_4^{\boldsymbol{\alpha}'}}(-1) \setminus M_4^{\boldsymbol{\alpha}}(-1)$ be the complement of $M_4^{\boldsymbol{\alpha}}(-1)$ in $\overline{M_4^{\boldsymbol{\alpha}'}}(-1)$. (Note that $\boldsymbol{\alpha} = \boldsymbol{\alpha}'/2$). Then $\mathcal{D}$ is a flat reduced divisor over $T_4 \times \Lambda_4$.*

3. *For each $(\mathbf{t}, \boldsymbol{\lambda})$, set*

$$\overline{S}_{\mathbf{t},\boldsymbol{\lambda}} := \overline{\pi}_4^{-1}(\mathbf{t}, \boldsymbol{\lambda}) := \overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1).$$

Then $\overline{S}_{\mathbf{t},\boldsymbol{\lambda}}$ is a smooth projective surface which can be obtained by blowing-ups at 8 points of the Hirzeburch surface $\mathbf{F}_2 = \mathrm{Proj}(\mathcal{O}_{\mathbf{P}^1}(-2) \oplus \mathcal{O}_{\mathbf{P}^1})$ of degree 2. The surface has a unique effective anti-canonical divisor $-K_{S_{\mathbf{t},\boldsymbol{\lambda}}} = \mathcal{Y}_{\mathbf{t},\boldsymbol{\lambda}}$ whose support is $\mathcal{D}_{\mathbf{t},\boldsymbol{\lambda}}$. Then the pair

$$(\overline{S}_{\mathbf{t},\boldsymbol{\lambda}}, \mathcal{Y}_{\mathbf{t},\boldsymbol{\lambda}}) \tag{10}$$

is an Okamoto-Painlevé pair of type $D_4^{(1)}$. That is, the anti-canonical divisor $\mathcal{Y}_{\mathbf{t},\boldsymbol{\lambda}}$ consists of 5-nodal rational curves whose configuration is same as Kodaira–Néron degenerate elliptic curves of type $D_4^{(1)}$ (=Kodaira type I_0^*). Moreover we have $(M_4^{\boldsymbol{\alpha}}(-1))_{\mathbf{t},\boldsymbol{\lambda}} = (\overline{M_4^{\boldsymbol{\alpha}'}}(-1))_{\mathbf{t},\boldsymbol{\lambda}} \setminus \mathcal{Y}_{\mathbf{t},\boldsymbol{\lambda}}$.

Okamoto Painlevé pair of type P_{VI}

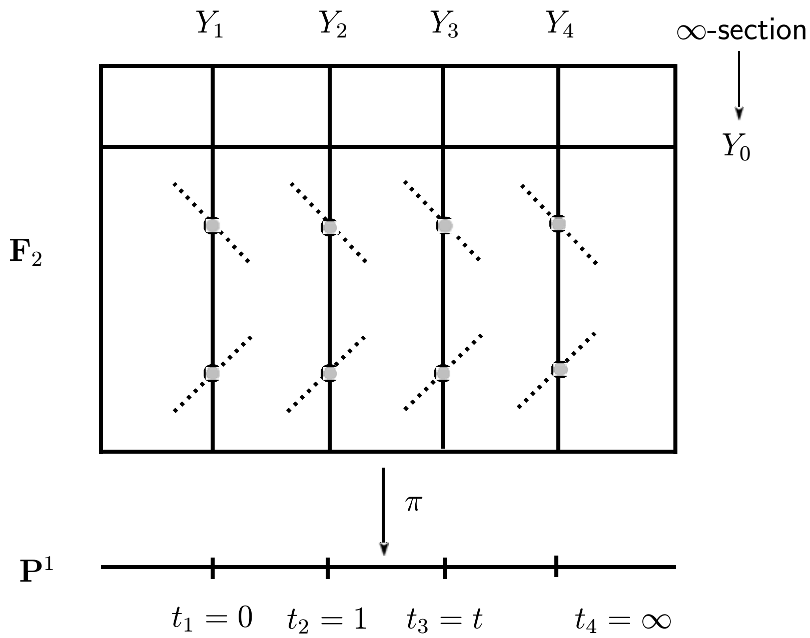


FIGURE 3. Okamoto-Painlevé pair of type $D_4^{(1)}$

Proposition 2. *The invariant ring* $(R_3)^{Ad(SL_2(\mathbf{C}))}$ *is generated by seven elements* $x_1, x_2, x_3, a_1, a_2, a_3, a_4$ *and there exist a relation*

$$f(\mathbf{x},\mathbf{a}) = x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1(\mathbf{a})x_1 - \theta_2(\mathbf{a})x_2 - \theta_3(\mathbf{a})x_3 + \theta_4(\mathbf{a}), \tag{11}$$

where we set

$$\begin{aligned} \theta_i(\mathbf{a}) &= a_ia_4 + a_ja_k, \quad (i,j,k) = a \text{ cyclic permutation of } (1,2,3), \\ \theta_4(\mathbf{a}) &= a_1a_2a_3a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4. \end{aligned}$$

Therefore we have an isomorphism

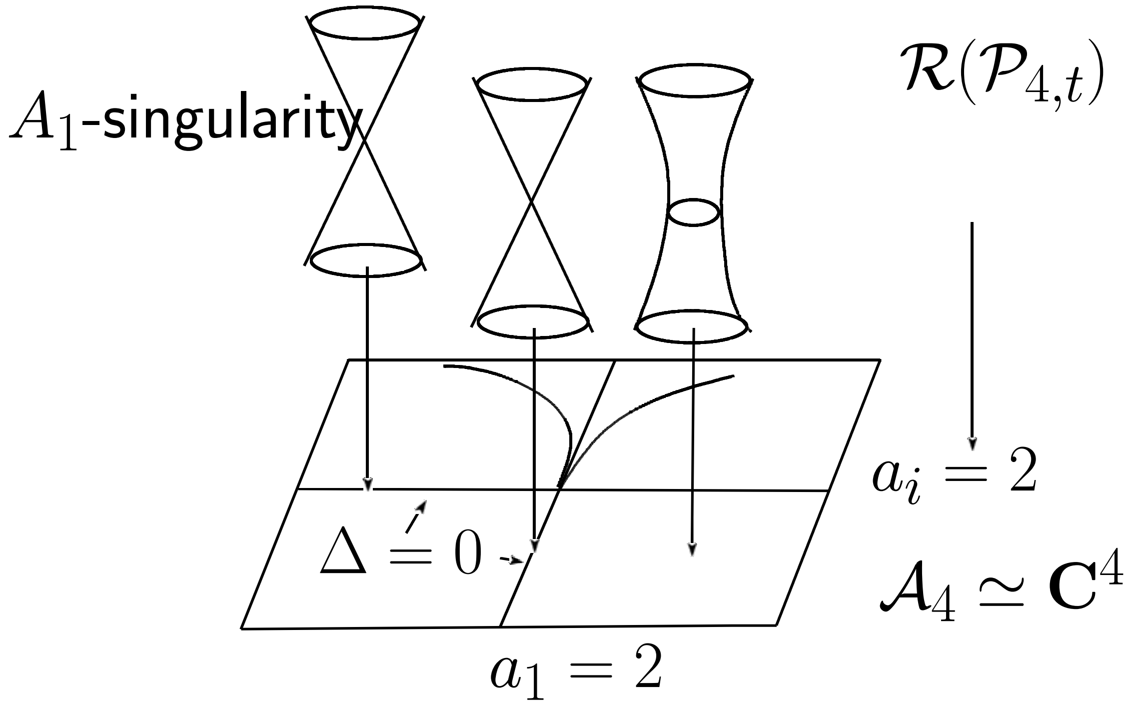
$$(R_3)^{Ad(SL_2(\mathbf{C}))} \simeq \mathbf{C}[x_1, x_2, x_3, a_1, a_2, a_3, a_4]/(f(\mathbf{z},\mathbf{a})).$$

Hence

$$\mathbf{Rep}(\mathbf{P}^1, (t_1, t_2, t_3, t_4), 2)_{\mathbf{a}} = \mathrm{Spec}\,(R_3)^{Ad(SL_2(\mathbf{C}))}$$

is isomorphic to an affine cubic.

$$\boxed{a_i = 2\cos 2\pi\lambda_i} \qquad \textit{for } 1 \le i \le 4. \tag{12}$$



The family of affine cubic surfaces

$$
\begin{tikzcd}
M_n^{\boldsymbol{\alpha}}(L) \arrow[r, "\mathbf{RH}_n"] \arrow[d, "\pi_n"'] & \mathcal{R}_n \arrow[d, "\phi_n"] \\
T_n' \times \Lambda_n \arrow[r, "(1\times\mu_n)"] & T_n' \times \mathcal{A}_n.
\end{tikzcd}
\tag{13}
$$

Here, we have $1 \times \mu_n$ $(1 \times \mu_n)(\mathbf{t}, \boldsymbol{\lambda}) = (\mathbf{t}, \mathbf{a})$

$$
\boxed{a_i = 2\cos 2\pi\lambda_i} \quad \text{for } 1 \leq i \leq n. \tag{14}
$$