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TOWARDS AN R-MATRIX FORMULATION FOR Q-DEFORMED INTEGRABLE SYSTEMS

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- AIM : to cast Classical (and hopefully Quantum) Hamiltonian Systems integrable by "coalgebra symmetry" approach in the more familiar Lax formalism
- TOOLS: usual tricks of soliton theory and a bit of guesswork: much has still to be done...

PLAN OF THE LECTURE

1. Reminder of Classical $\mathcal{U}_q(sl(2))$ Gaudin Hamiltonian and cluster variables
2. Lax pair for Classical $\mathcal{U}_q(sl(2))$ Gaudin model

I. Classical $\mathcal{U}_q(sl(2))$ Gaudin Hamiltonian and cluster variables

Start from the *standard* q-deformed $sl(2)$, defined by Poisson brackets:

$$\{X_3, X_\pm\} = \pm 2 X_\pm \,, \qquad \{X_+, X_-\} = \frac{\sinh(z X_3)}{z}$$

with Casimir function:

$$\mathcal{C}_z = \frac{\sinh^2\left(\frac{z}{2} X_3\right)}{z^2} + X_+ X_- \,.$$

and admissible coproduct map:

$$\Delta(X_3) = X_3 \otimes I + I \otimes X_3$$
$$\Delta(X_\pm) = X_\pm \otimes e^{\frac{z}{2} X_3} + e^{-\frac{z}{2} X_3} \otimes X_\pm \,.$$

Choosing the function $\mathcal{H} \doteq \Delta^{(N)}(\mathcal{C}_z)$ as our N–body Hamiltonian, the equations of motions are easily written (and solved in terms of elementary functions) in the alternative basis:

$$S_3 \doteq X_3\,, \qquad S_\pm \doteq e^{-\frac{z}{2}X_3} X_\pm\,,$$

with Poisson brackets

$$\{S_3, S_\pm\} = \pm 2 S_\pm\,, \qquad \{S_+, S_-\} = \frac{1 - e^{-2zS_3}}{2z} + 2z S_+ S_-\,,$$

Casimir

$$\mathcal{C}_z = \frac{\sinh^2\left(\frac{z}{2}S_3\right)}{z^2} + e^{zS_3} S_+ S_-$$

and comultiplication

$$\Delta(S_3) = S_3 \otimes I + I \otimes S_3$$
$$\Delta(S_\pm) = S_\pm \otimes I + e^{-zS_3} \otimes S_\pm\,.$$

They read ($S_i^{(m)} \doteq \Delta^{(m)}(S_i)$):

$$\dot{S}_3^{(m)} = 2e^{z\delta_3}\left(S_+^{(m)}\delta_- - S_-^{(m)}\delta_+\right),$$

$$\dot{S}_{\pm}^{(m)} = \pm 2z\delta_{\mp}\, S_{\pm}^{(m)} \left(\delta_{\pm} - S_{\pm}^{(m)}\right) \mp \frac{\sinh(zS_3)}{z}\, S_{\pm}^{(m)} \pm \delta_{\pm}\, e^{z\delta_3}\frac{1-e^{-2zS_3^{(m)}}}{2z}\,,$$

where $\delta_i \doteq \Delta^{(N)}(S_i)$.

Hence, the cluster variables yielded by the partial coproducts of the generators are in fact *separation* variables. While in the non-deformed case the same goal (separation and solution) can be achieved either by single-particle variables or by cluster variables, only the cluster variables do the job in the deformed case. However, *one has to pay a nonnegligible price*: **the Poisson brackets are no-more ultralocal.**

II. Lax pair for Classical $\mathcal{U}_q(sl(2))$ Gaudin model

To get a Lax pair in the original basis, we make the tentative choice:

$$L^{(m)} \doteq \begin{pmatrix} a^{(m)} & b^{(m)} \\ c^{(m)} & -a^{(m)} \end{pmatrix},$$

where:

$$a^{(m)} = \frac{\sinh\left(\frac{z}{2}X_3^{(m)}\right)}{z},$$
$$b^{(m)} = X_-^{(m)}, \qquad c^{(m)} = X_+^{(m)}.$$

Remark

$$\mathrm{Tr}\left(L^{(m)}\right)^2 = \Delta^{(m)}(\mathcal{C}_z)$$

gives the m–th integral of motion.

Theorem (easy to show): **The evolution equation shown in the previous transparency admit the Lax representation**:

$$\frac{dL^{(m)}}{dt} = \left\{L^{(m)}, \mathcal{H}\right\} = \left[L^{(m)}, M^{(m,N)}\right],$$

with

$$M^{(m,N)} \doteq \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix},$$

where

$$\begin{aligned}
\alpha &= -\frac{\sinh(z\lambda_3)}{z} + 2z\lambda_+\lambda_- - ze^{\frac{z}{2}\lambda_3}e^{-\frac{z}{2}X_3}(c\lambda_- + b\lambda_+)\,, \\
\beta &= -\frac{1}{2}\lambda_-\left(1 + e^{-zX_3}\right)\, e^{\frac{z}{2}\lambda_3}\,, \\
\gamma &= -\frac{1}{2}\lambda_+\left(1 + e^{-zX_3}\right)\, e^{\frac{z}{2}\lambda_3}\,.
\end{aligned}$$

and

$$\lambda_3 \doteq \Delta^{(N)}(X_3), \qquad \lambda_\pm \doteq \Delta^{(N)}(X_\pm)$$

.

The (m) superscript has been omitted.

So far, we have a Lax pair for each cluster variable. A "local" (i.e. for a given cluster) $R-$matrix is easily calculated. Indeed it holds ($\forall m$, that will then be omitted):

$$\{L_{ij}, L_{kl}\} = [R, L \otimes I + I \otimes L]_{ij,kl}$$

R being the *dynamical* $R-$matrix (Π is the usual permutation operator) :

$$R \;=\; \cosh(zX_3/2)\Pi$$

A global Lax pair can be obtained by adding together all partial $L^{(m)}$ matrices in block–diagonal form:

$$L(\lambda) == \bigoplus_{j=1}^{N} \lambda^{1-j} L^{(j)}\,, \qquad M = \bigoplus_{j=1}^{N} M^{(j)}\,.$$

The generating function of the integrals of motion is

$$\mathrm{Tr} L(\lambda)^2 = \sum_{j=1}^{N-1} \lambda^{2\,(1-j)} \Delta^{(j)}(\mathcal{C}_z)\ ,.$$

However, due to the nonlocal Poisson structure:

$$\{L^m, L^n\} \neq 0, m \neq n$$

finding an R-matrix formulation for such a "global" Lax matrix is a nontrivial task.

Also, the Lax representation has to be extended to the Quantum case.

And finally: we still don't know whether there exist integrable models of Gaudin type (hence, long-range) which combine $q-$symmetry and inhomogeneities.