Asymptotics of PIII transcendent in the complex plane

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Outline

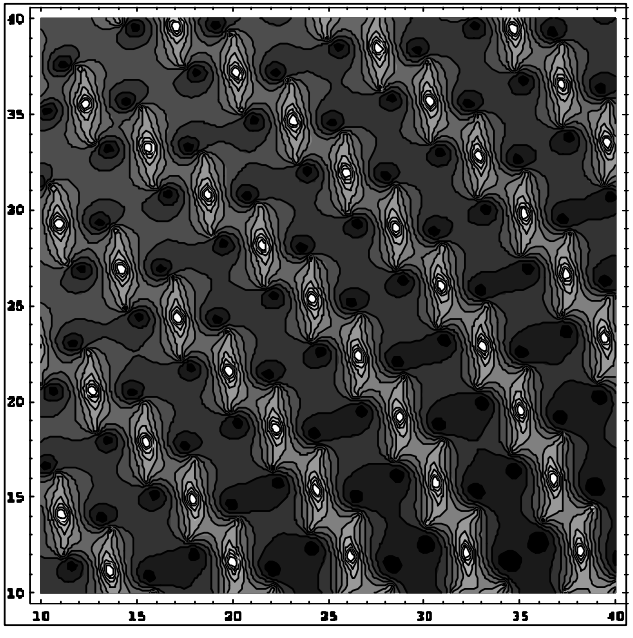
- What is the *Boutroux ansatz* for PIII?
- Whitham averaging and exact solutions
- Isomonodromic deformations and matrix Riemann-Hilbert problem
- Asymptotic ‘undressing’ of the RH problem
- Justification of the Boutroux ansatz

Special (and most popular) case of PIII equation

$$u_{xx} + \frac{1}{x} u_x + \sin u = 0, \qquad x, u \in \mathbb{C}$$

with initial condition $u(x) = a \ln x + b + O(x^{2-\mathrm{Im}\, a}), \quad x \to 0$

How to get ***uniform*** asymptotic expansion in the sector of the complex plane $x = re^{ij}, \ 0 < j < p/2, \ r >> 1$



⇦ Numeric simulation of the solution:
° - poles, ? - zeroes

Why the lattice of poles and zeroes?

(Answer: elliptic function as the leading term of asymptotics)

First a naïve way: Whitham averaging of PIII itself

$$u_x^2 - 2\cos u = -\int \frac{2u_x^2}{x} dx,$$

$$u = U + o(1),$$

where U is elliptic function:

$$U_x^2 - 2\cos U = 2F, \qquad F = const,$$

$$U(x_0) = u(x_0), \quad U_x(x_0) = u_x(x_0)$$

$$\sin\left(\frac{U(x) - U_0}{2}\right) = \mathrm{sn}\left((x - x_0)\sqrt{2(F+1)}\,\middle|\,\sqrt{\tfrac{2}{F+1}}\right),$$

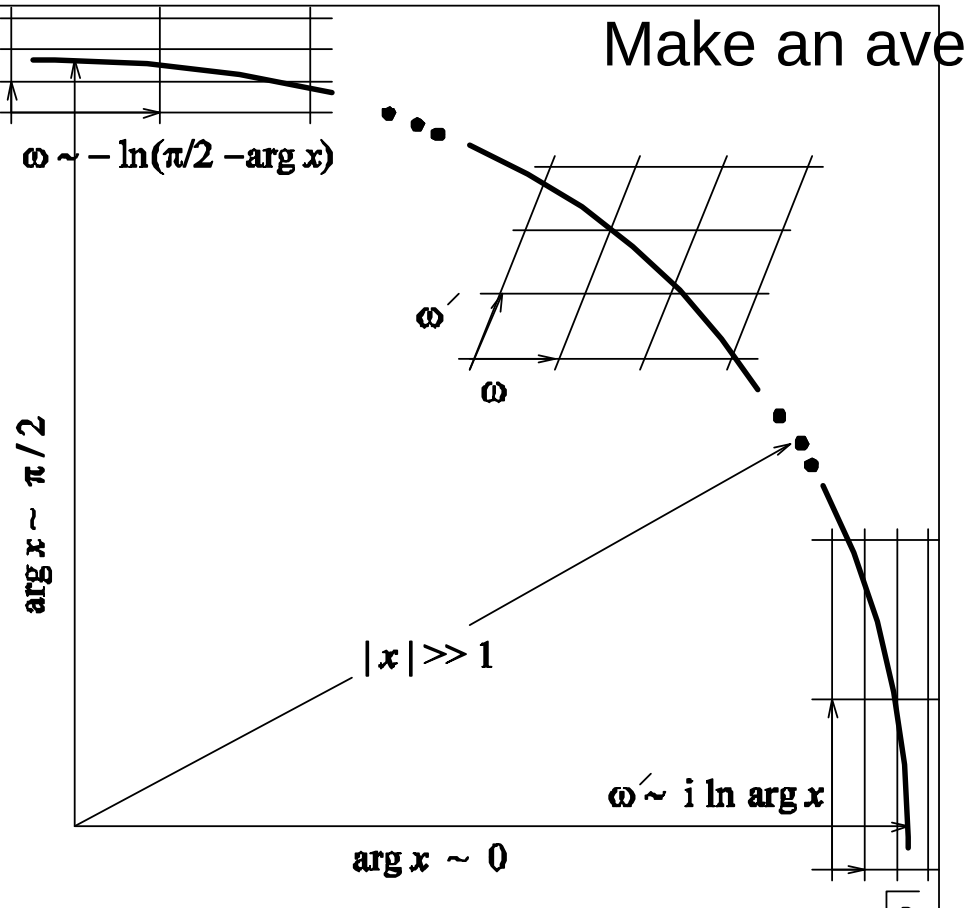
sn - Jacobi sine

Here $F = -\int\limits_{\text{period of } U} \frac{U_x^2}{x}\, dx$ (full energy of the pendulum),

and approximation $u = U + o(1)$ holds for $|x - x_0| < O(1)$

Boutroux ansatz: $U_x^2 - 2\cos U = 2F(j), \qquad j = \arg x, \quad x >> 1$

and choose a proper averaging of the r.h.s. $F = -\int\limits_{\text{period of } U} \frac{U_x^2}{x}\,dx$



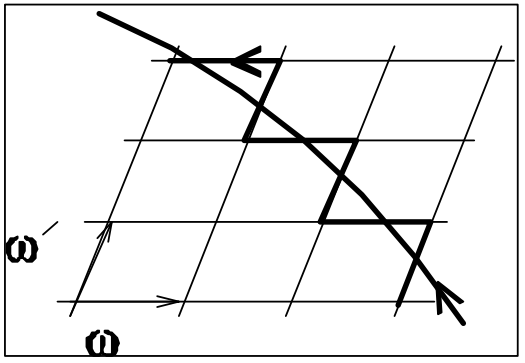
Make an averaging along the *arc* |x| = *r*

$$\int\limits_{|x|=r} \frac{U_x^2}{x}dx = i\int\limits_{j_0}^{j} U_x^2\left(re^{ij}\right)dj$$

$$= i\int\limits_{j_0}^{j} \left\langle U_x^2\left(re^{ij}\right)\right\rangle dj + o(1)$$

Averaging of two-periodic function

$$\left\langle U_x^2 \right\rangle = \frac{1}{|\ell|}\int_{\ell} 2\left(F+\cos U\right)dx = \frac{\int_{\ell}\sqrt{2\left(F+\cos U\right)}dU}{\int_{\ell}\frac{dU}{\sqrt{2\left(F+\cos U\right)}}}$$



$$\ell = nw + mw',$$

$$n, m >> 1$$

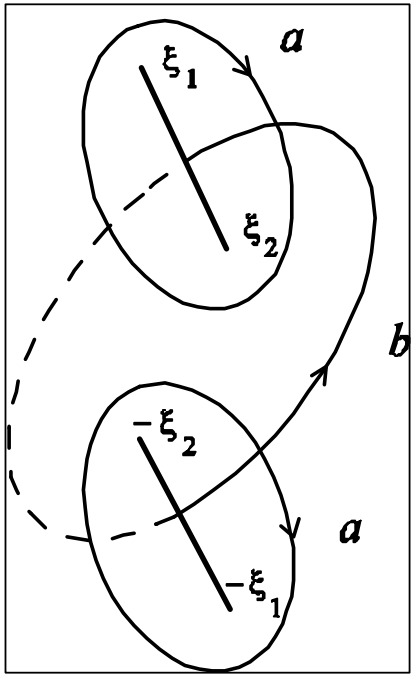
Whitham equation

$$F(j) = -i\int_{j_0}^{j}\frac{G(F)}{G'(F)}dj$$

where $$G(F) = \int_{\ell}\sqrt{2\left(F+\cos U\right)}dU$$

Solving the Whitham equation

$$F_j = \frac{-i}{(\ln G)_j} \Rightarrow \quad (\ln G)_j = -i \Rightarrow \quad G(F) = e^{-i(j-j_0)}$$



basic cycles of $\sqrt{F+\cos U}$

$$\arg G_a(F) = \arg G_b(F) = -j + j_0$$

$$G_{a,b}(F) = \int_{a,b} \sqrt{2\left(F+\cos U\right)}\,dU$$

$$G_a = 2\sqrt{2}\sqrt{F+1}\,E(k),$$

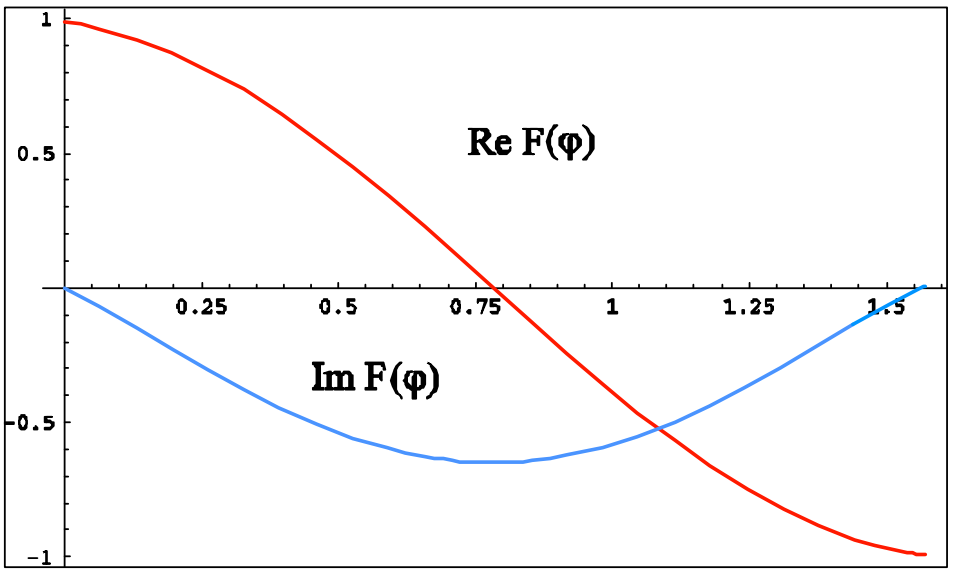
$$G_b = \frac{\sqrt{2}i}{\sqrt{F+1}}\left(K'(k) - E'(k)\right),$$

$$k = \frac{\sqrt{2}}{\sqrt{F+1}}$$

Uniformi-zation of complete elliptic integrals

Solving the Whitham equation (continued)

$$\frac{dF}{dj} = -2i(F+1)\frac{E\left(\sqrt{\frac{2}{F+1}}\right)}{K\left(\sqrt{\frac{2}{F+1}}\right)},$$



Numeric solution
by ‘Mathematica’

Boundary condition:

$$F(0) = 1,$$

$$U(x)\Big|_{F=1} = \arcsin e^{-x},$$

$$w = 2p,\ w' = \infty$$

This yields

$$F\left(\frac{p}{2}\right) = -1,$$

$$U(x)\Big|_{F=-1} = \arcsin e^{ix},$$

$$w = \infty,\ w' = 2p$$

How to find the phase $x_0(j)$ of the elliptic ansatz?

$$u(x) = U(x - x_0(j) \mid F(j)) + o(1), \qquad |x| >> 1$$

Answer: integrability of PIII and Isomonodromic Method

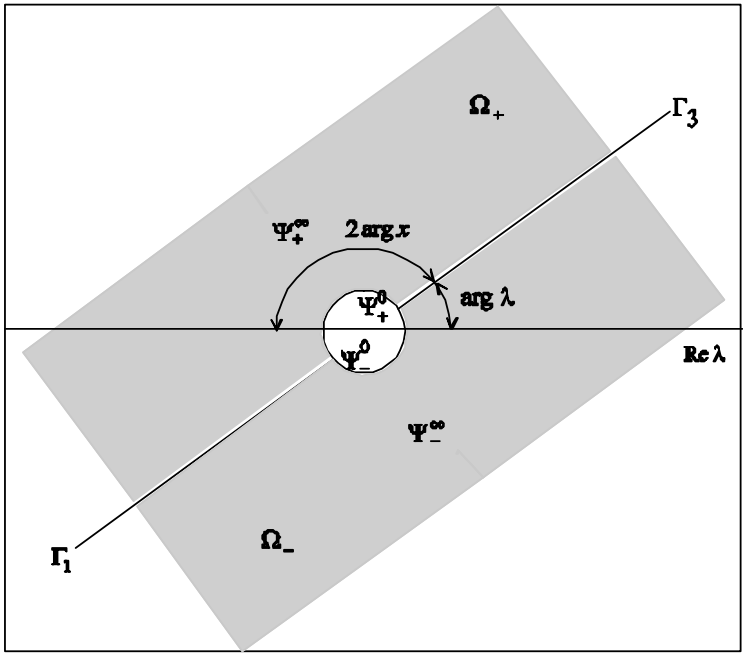
$$\Psi_x = r^2 \begin{pmatrix} -\frac{iX^2}{16} - \frac{i\cos u}{x^2} & -\frac{iXu'}{4x} + \frac{i\sin u}{x^2} \\ -\frac{iXu'}{4x} - \frac{i\sin u}{x^2} & \frac{iX^2}{16} - \frac{i\cos u}{x^2} \end{pmatrix} \Psi, \quad \Psi_x = \begin{pmatrix} -\frac{iXx}{8} & -\frac{iu'}{2} \\ -\frac{iu'}{2} & \frac{iXx}{8} \end{pmatrix} \Psi$$

$$r^{-1}L_x - A_x + [L, A] = 0 \quad \Leftrightarrow \quad u'' + r^{-1}\frac{u'}{X} + \sin u = 0$$

where $X = \frac{x}{r} = e^{ij}, \quad r >> 1, \quad u' = u_x$

- Direct monodromy problem for $\Psi \Leftrightarrow$ initial condition for u
- Apply elliptic ansatz U in the Lax pair
- Find uniform asymptotics of Ψ as $r >> 1$

Riemann-Hilbert problem



$$\Psi_{\pm}^{\infty} \to \exp\left(-\frac{irX^2}{16}xs_3\right), \quad x \to \infty, \quad x \in \Gamma_1, \Gamma_3,$$

$$\Psi_{\pm}^{0} \to \begin{pmatrix} \cos\frac{u}{2} & i\sin\frac{u}{2} \\ i\sin\frac{u}{2} & \cos\frac{u}{2} \end{pmatrix} \exp\left(-\frac{ir^2}{x}s_3\right), \quad \begin{matrix} x \to 0, \\ \operatorname{Im} x \gtrless 0 \end{matrix}$$

$$\Psi_{-}^{\infty} = \Psi_{+}^{\infty} S_j, \quad x \in \Gamma_j, \quad j = 1,3$$

$$\Psi_{\pm}^{0} = \Psi_{\pm}^{\infty} Q, \quad x \in \Omega_{\pm}.$$

$$S_1 = \begin{pmatrix} 1 & q \\ p & 1+pq \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & -p \\ -q & 1+pq \end{pmatrix},$$

$$Q = \frac{1}{\sqrt{1+pq}} \begin{pmatrix} 1 & p \\ -q & 1 \end{pmatrix}$$

p and q - monodromy data (first integrals of PIII), independent of x

Direct monodromy problem. WKB solutions

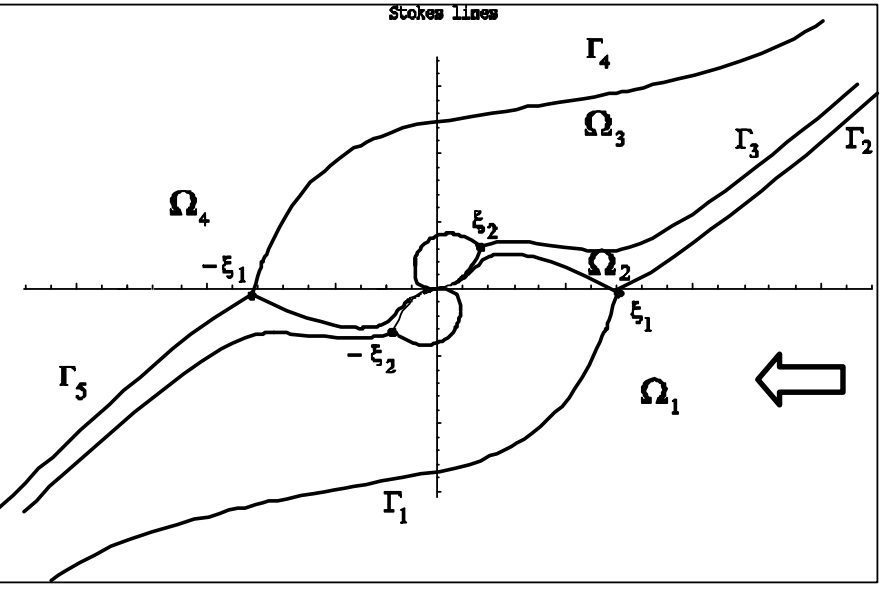
$$\Psi_x = r^2 \begin{pmatrix} -\frac{iX^2}{16} - \frac{i\cos U}{x^2} & -\frac{iX\sqrt{2(F+\cos U)}}{4x} + \frac{i\sin U}{x^2} \\ -\frac{iX\sqrt{2(F+\cos U)}}{4x} - \frac{i\sin U}{x^2} & \frac{iX^2}{16} - \frac{i\cos U}{x^2} \end{pmatrix} \Psi, \qquad A$$

$$\det A = m(x) = \frac{X^2}{16x^2}\sqrt{x^4 + \frac{32F}{X^2}x^2 + \frac{256}{X^4}} = \frac{X^2}{16x^2}\sqrt{(x^2 - x_1^2)(x^2 - x_2^2)}.$$

$$\Psi_j^{WKB} = \left[T + O(r^{-1}m^{-3/2})\right] \exp\left[-i\int_{x_j}^{x} (xms_3 - T^{-1}D_z T)dz\right] C,$$

$$T = \sqrt{\frac{d_1}{m}} \begin{pmatrix} 1 & \frac{d_1 - m}{d_2} \\ \frac{m - d_1}{d_2} & 1 \end{pmatrix}, \quad d_1 = \frac{X^2}{16} - x^{-2}, \quad d_2 = \frac{X\sqrt{2F + 2\cos U}}{4x},$$

WKB solutions (continued)

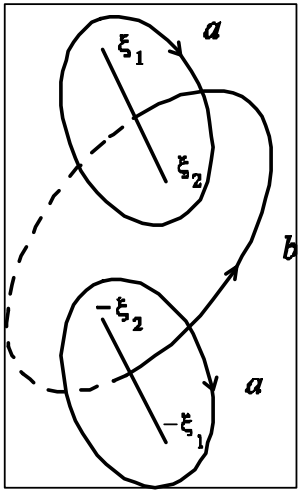


Stokes lines: $\mathrm{Re}\left[ir\int_{x_j}^{x} m(z)dz \right] = 0$

$\pm x_1, \pm x_2$ - zeroes of $m(x)$

⇐ Stokes graph

Basis of cycles ⇒



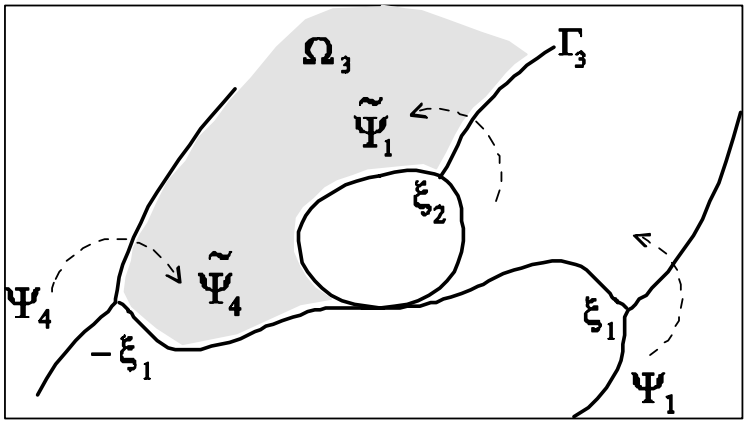
Monodromy equation:

$$\Psi_1 = \Psi_4 \begin{pmatrix} 1 & q \\ p & 1+pq \end{pmatrix}$$

$$\Psi_1^{WKB} = \left[T + O(r^{-1}m^{-3/2}) \right] \exp\left[-ixs_3 \int_{x_1}^{x} m(z)dz - \frac{1}{4}\oint_a m(z)dz \right], \quad x \in \Omega_1$$

$$\Psi_4^{WKB} = \left[T + O(r^{-1}m^{-3/2}) \right] \exp\left[-ixs_3 \int_{-x_1}^{x} m(z)dz + \frac{1}{4}\oint_{2a-b} m(z)dz \right], \quad x \in \Omega_4$$

How to transfer Ψ_1, Ψ_4 into a common domain?



$$m = 6g_j r^{-1/3} z^{1/2} + O((x - x_j)^{3/2})$$

$$z = r^{2/3}(x - x_j),$$

$$\Psi = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\begin{cases} c_{1,x} = -\dfrac{irX^2}{16x^2}(x - x_1)(x - x_2)c_2 \\ c_{2,x} = -\dfrac{irX^2}{16x^2}(x + x_1)(x + x_2)c_1, \end{cases} \Rightarrow \begin{cases} c_1 = 0, \\ c_{2,zz} + 36g_j^2 z c_2 = 0, \end{cases}$$

$c_2(z)$ - Airy function with normalizations

$$c_2(z) = z^{-1/4}\left[1 + O(z^{-3/2})\right]\exp\left(-4igz^{3/2} - \frac{pi}{4}\right),$$

$$\overline{c_2}(z) = z^{-1/4}\left[1 + O(z^{-3/2})\right]\exp\left(4igz^{3/2} + \frac{pi}{4}\right),$$

$$z \to \infty e^{-2g/3}.$$

Whitham equations revisited

Then in the common domain Ω_3 we have

$$\Psi_{1,4}(x)=\sqrt{6}T\left[1+O(r^{-1}m^{-3/2})\right]\exp\left[-ixs_3\int_{x_{1,2}}^{x} m(z)dz\right]\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}D_j, \quad j=1,4,$$

where $D_1=\exp\left[\frac{ix}{4}\oint_a m(z)\,dz\right]$, $D_4=\exp\left[\frac{ix}{4}\oint_{2a-b} m(z)\,dz\right]$,

Since $\Psi_4\Psi_1^{-1}$ is finite in $|x|>>1$, this yields the Whitham equations

$$\arg\left[\oint_a m(z)\,dz\right]=\arg\left[\oint_b m(z)\,dz\right]=-\arg x=-j$$

They are equivalent to deformation equations obtained above

$$\arg\int_a\sqrt{2\left(F(j)+\cos U\right)}dU=\arg\int_b\sqrt{2\left(F(j)+\cos U\right)}dU=-j$$

Zeroes of the elliptic function ansatz

Consider $x_{m,n}$ such that $\cos U = 1$

$$\int m(z)\,dz = -\frac{X_{m,n}}{4}\int \frac{1}{n^2}\sqrt{n^4 + Fn^2 + 1}\,dn, \quad F = F_0 + \frac{F_1}{r} + O(r^{-2}).$$

$$\Psi_4^{-1}\Psi_1 = D_4^{-1}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\exp\left[ixs_3 \int_{x_2}^{x_1} m(z)\,dz \right]\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} D_1 = \begin{pmatrix} 1 & q \\ p & 1+pq \end{pmatrix}.$$

$$rX_{m,n} = mw + nw' + x_0 + O(r^{-1}), \qquad w = 2kK, \quad w' = ikK'.$$

$$\begin{cases} \log(-p) = \dfrac{ix_0}{2k}(K' - E' + E) + 2k(K + K')F_1, \\ \log(-1-pq) = \dfrac{ix_0}{2k}(K' - E') + 2kKF_1. \end{cases}$$

$$x_0(f) = \frac{ik(f)}{2p}\left[K(k)\log(-p) - \left(K(k) - iK'(k)\right)\log(-1-pq)\right], \quad k^2(j) = \frac{2}{F(j)+1}.$$

Final result

Theorem Given p and q – the monodromy data parametrizing PIII

$$u_{xx} + \frac{1}{x} u_x + \sin u = 0, \qquad x, u \in \mathbb{C}$$

the solution has the asymptotics as $x = re^{ij}$, $0 < j < \frac{p}{2}$, $r >> 1$

$$u(x) = 2\arcsin\left\{ \mathrm{sn}\left(\frac{2(x - x_0(j))}{k(j)} \middle| k(j) \right) \right\} + o(1),$$

$$x_0(f) = \frac{ik(f)}{2p}\left[K(k)\log(-p) - \left(K(k) - iK'(k)\right)\log(-1 - pq) \right], \quad k^2(j) = \frac{2}{F(j)+1}.$$

$$K = K(k) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}, \quad K' = K(k'),$$

$$E = E(k) = \int_0^1 \sqrt{\frac{1-k^2z^2}{1-z^2}}\, dz, \quad E' = E(k'),$$

$$\arg \int_{a,b} \sqrt{2\left(F(j) + \cos U\right)}\, dU = -j$$

Monodromy data and initial condition

For the PIII equation $u_{xx}+\frac{1}{x}u_x+\sin u=0, \quad x,u\in\mathbb{C}$

with initial condition $u(x)=a\ln x+b+O(x^{2-\mathrm{Im}\,a}), \quad x\to 0, \quad |\mathrm{Im}\,a|<2$

the monodromy data is given explicitly*

$$p=\frac{Ae^{-\frac{\pi a}{4}}-Be^{\frac{\pi a}{4}}}{A+B}, \quad q=\frac{Be^{-\frac{\pi a}{4}}-Ae^{\frac{\pi a}{4}}}{A+B},$$

where $A=2^{\frac{3ia}{2}}e^{\frac{ib}{2}}\Gamma^2\left(\frac{1}{2}+\frac{ia}{4}\right), \quad B=2^{-\frac{3ia}{2}}e^{-\frac{ib}{2}}\Gamma^2\left(\frac{1}{2}-\frac{ia}{4}\right)$

and Γ is gamma-function.

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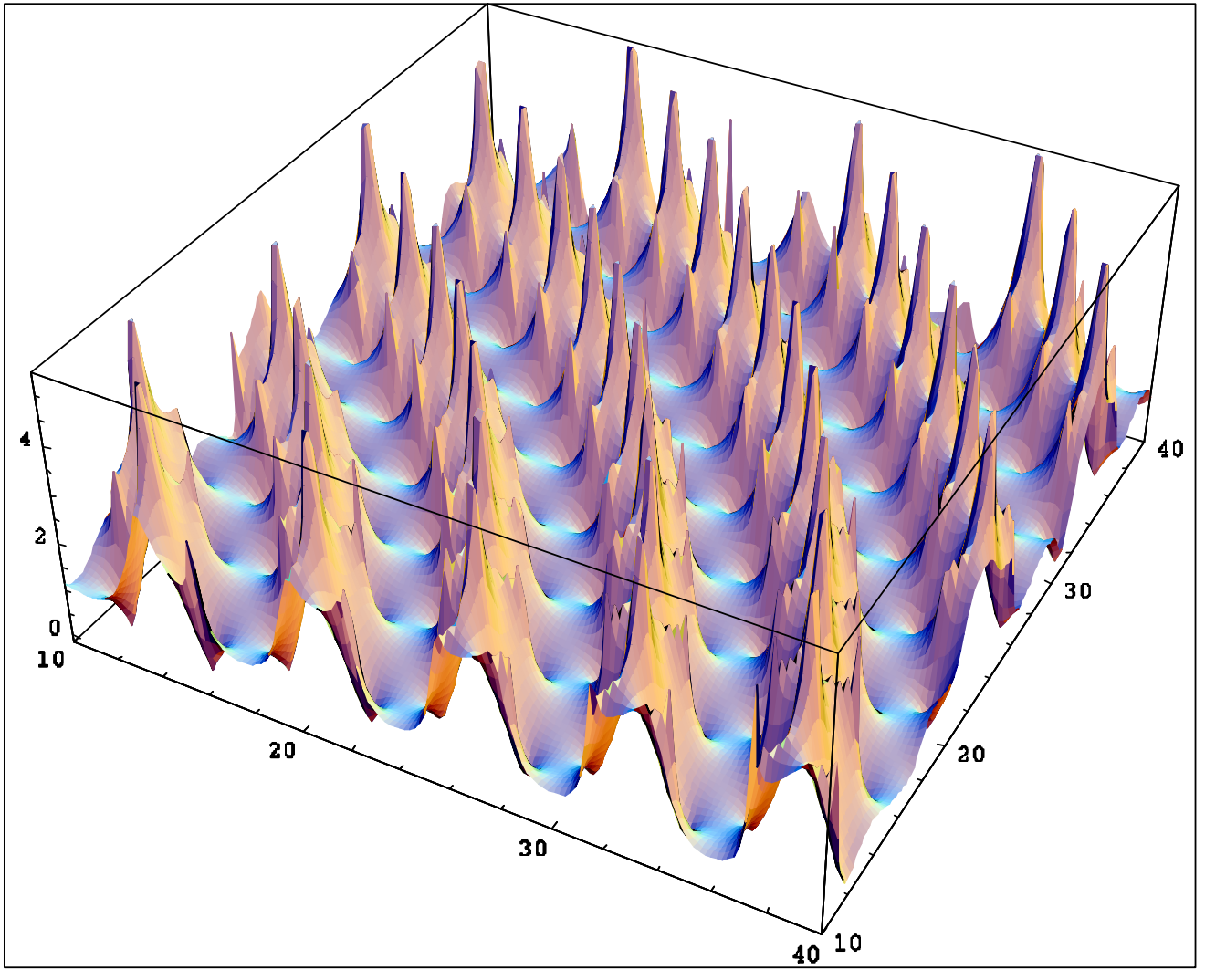
* V.Yu.Novokshenov, A.G.Shagalov, Bound states of the elliptic sine-Gordon equation,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