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Topology of The Generic Hamiltonian System of the Riemann Surfaces

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Riemann Surface Γ, closed 1-form ω

$\omega = 0 \Leftrightarrow$ Hamiltonian Foliation

genus $g = 0$ trivial

— " — $g = 1$ some nontrivial ergodic theory (Sinai -Khanin)

$g=1$: $\omega = Re(V)$, V - holomorphic; a straight-line flow

$g>1$: $\omega = Re(V)$, V - holomorphic; higher genus analogs of the straight-line flow.

<u>Remark</u>: <u>another interesting example is coming from the quantum Solid State Physic of Metals</u>

$\mathrm{II} \subset T^3$ (3-torus of quasimomenta)

ω = Restiction of <u>constant</u> 1-form on II (motion of electrons in <u>lattice</u> and <u>magnetic field</u> /
<u>S.N., A. Zorich, I. Dynnikov, S. Tsarev, A. Maltrev, 1982-2004</u>

These closed 1-forms are nongeneric; systems can be chaotic for the directions of magnetic field covering the set whose fractal dimension is less than 1 in S^2;

<u>Normally they generate systems reducible to genus 1 behavior.</u>

It leads to some important physical consequences for electrical conductivity.

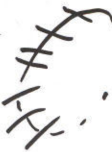
Our goal today is to study generic Hamiltonian System given by the real part of holomorphic 1-form on Π:

$$\Pi:\ y^2 = P_{2n+2}(x),\quad \omega = \mathrm{Re}\left[\frac{a_0 + \dots + a_{n-1}x^{n-1}}{\sqrt{P_{n+2}(x)}}\,dx\right]$$

Properties:

1. $2n-2$ saddles
2. No separatrix connections
3. Every periodic trajectory is homologous to zero. We assume that there are no periodic orbits.

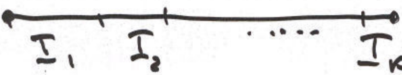
Classical approach (since 1960s): Take any closed transversal curve $\gamma \subset \Pi$. It defines a "Poincaré Map" $\gamma \to \gamma$



except finite number of points. This map is an Isometry, so everything is determined by these points

$S^1 = I_1 \cup I_2 \cup \dots \cup I_K$

and permutation $\sigma \in S_K$



$(1,\dots,K) \mapsto (i_1,\dots,i_K)$

(Big ergodic theory exists here)

Qualitative Theory of 2D Dynamical systems is based on transversal curve since Poincaré:

Poincaré type for $g=0$

(piece of trajectory + small transversal)

$g=1$: reduction of the flow to the map $S^1 \to S^1$

Question: How to describe all closed transversal curve for the Hamiltonial Flow on $\mathbb{II}$ for $g>1$?

Almost transversal curves = pieces of trajectories + positively oriented transversal pieces (like Poincaré type

Homotopy classe of the closed positively oriented transversal (almost transversal) curves passing through the point $x_0 \in \mathbb{I}$

$\gamma_1 \circ \gamma_2$

$\pi^+(\mathcal{R}, x_0) \Longrightarrow$

$\Longrightarrow \pi_1(\mathbb{I}, x_0)$

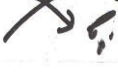
They form a

<u>Semigroup π^+</u>

<u>How to calculate this semigroup?</u>

"Transversal Canonical Basis" (TCB) =

= 2n closed transversal curves $a_1, b_1, \dots, a_n, b_n \subset \mathbb{I}$ such that all of them are simple (smooth), and only nontrivial crossings are transv.

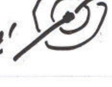
$a_i \circ b_i = 1$ point 

Theorem 1 (S.N. - G. Levitt).

For every generic hamiltonian foliation given by the real part of holomorphic 1-form there exists a TCB

Theorem 2 (S.N.) Every such system (foliation) can be presented gluing following pieces:

(1.) 2-tori $T^2_1, \dots, T^2_n$ with straight line flows and segments $s_1, \dots, s_n$, $s_j \subset T^2_j$, transversal to foliation

(2) Hamiltonian system on S^2 with generic hamiltonian $h: S^2 \to R$ and n pieces $t_1, \dots, t_n \subset S^2$ transversal to the flow and not crossing each other except that exactly two of them meet each other in every center 

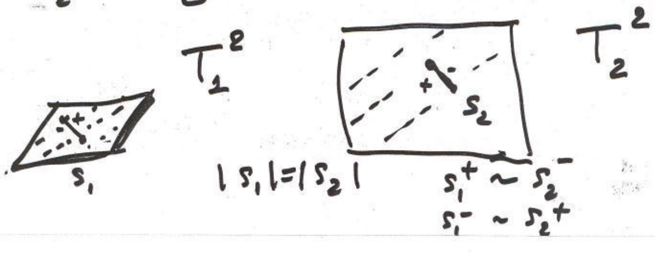
.

Every trajectory on S^2 meets at least one segment t_j (except saddles). For the transversal measure we have

$$|s_j| = |t_j|$$

Making cuts along t_j and s_j and identifying their boundaries (preserving transversal measures), we obtain our flow.

For $g=2$ sphere S^2 is not needed: we obtain flow identifying two 2-tori T_1^2, T_2^2 along the segments $s_1 \subset T_1^2$, $s_2 \subset T_2^2$

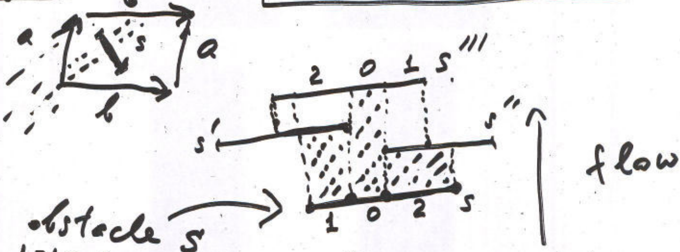


$|s_1| = |s_2|$

$s_1^+ \sim s_2^-$
$s_1^- \sim s_2^+$

Most important element of construction:

Torus T^2 with straight-line flow, a pair transversal cycles $a, b \in H_1(T^2)$ and transversal segment s: $|s| < |a| + |b|$



$|s| = 1 + 0 + 2$

Lemma. Every obstacle and flow define a "3-street" fundamental domain in R^2 for the group $Z \times Z$ with $\underbrace{\text{generators}}_{R^2 \to R^2}$ $a' : s \to s'$, $b' : s \to s''$, $a'b' : s \to s'''$

Let $b < a$ $s < a + b$;

Find sequence of integers s.t. $0 < a - l_1 b < \max(b, s)$, $a = a_0$, $b = b_0$

$a_1 = a - l_1 b$, $b_1 = b - l_2 a_1$

.... until difference is $< s$.

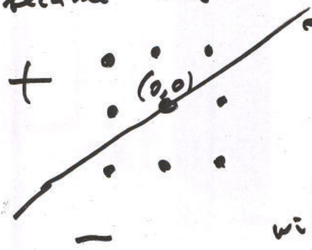
Let
$$a/b = l_1^* + \cfrac{1}{l_2^* + \cfrac{1}{l_3^* + \cdots}}$$
a Continued Fraction. We have
$$l_j^* = l_j, \quad j = 1, \ldots, m-1$$
$$0 < l_m < l_m^*, \quad l_{m+1} = 0$$
Consider transformations of free group $F_2\{u,v\}$ ~~also~~
$$\left.\begin{array}{l} u \to uv \\ v \to v \end{array}\right| T_1$$
$$\left.\begin{array}{l} u \to u \\ v \to vu \end{array}\right| T_2$$
Let
$$T = T_1^{l_1} T_2^{l_2} \cdots T_m^{l_m}$$
$$(T_m = T_1 \text{ or } T_2)$$
<u>Theorem</u>. There is a connection
$$(a, b) = T(a', b')$$
All transversal ~~Bases~~ in the torus T^2 with flow and obstruction S can be obtained from (a', b') in that way

3) Semigroup of positive closed transversal curves in the torus T^2 not crossing the obstacle S is generated by the elements a', b', s.t.

$$|a'|+|b'|>|S|,$$
$$|a'|<S,\quad |b'|<S$$

<u>Remark</u>. For $|S|=0$ this semigroup became infinitely generated.

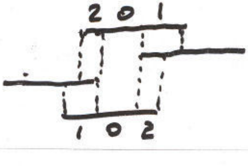
← direction of flow



For any small $|S|>0$ it is a free semigroup with 2 generators (a', b')

Sizes of the streets:

$$\left.\begin{aligned}|a'|&=1+0\\ |b'|&=0+2\end{aligned}\right\}\ |a'|+|b'|=\underbrace{(1+0+2)}_{=S}+0$$



Genus $g=2$:

$$\mathbb{T} = T_1^2 \cup_{S^1} T_2^2$$

Homological coding of trajectories

$$[\gamma] \in H_1(\mathbb{T}, S^1; \mathbb{Z}) = H_1(\mathbb{T}, \mathbb{Z}) = \mathbb{Z}^4$$

Each piece within torus T_j^2, $j=1,2$, define class

$$\lambda_q(\gamma) \in H_1(T_j^2; \mathbb{Z})$$

$$q \in \mathbb{Z},\ j = q \text{ modulo } 2$$

$$[\dots, \lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \dots] = \lambda(\gamma)$$

λ_q is equal to one of 3 elements $a'(s)$, $b'(s)$, $a'+b'(s)$ corr. to 3-street picture above in the torus $T^2_{j(q)} \setminus S^1$

<u>Remark</u>. Homological coding is well-defined for every genus g, but for $g=2$ we are aware that $\lambda_g(\gamma)$ belongs to $T^2_{j}(g)$ where $j = g$ modulo 2. <u>For genus >2 the sequence of tori became also random;</u> <u>for $g=2$ we have:</u> ..., 1, 2, 1, 2, 1, 2, ... <u>for these numbers</u>

It allows to define <u>NONABELIAN coding</u>, of γ because every <u>2-street passage</u> (1, 2) defines correctly the element of

$$\mu_Q(\gamma) \in \pi_1(\mathrm{II}, S^+) = \pi_1(\mathrm{II}, \text{point})$$

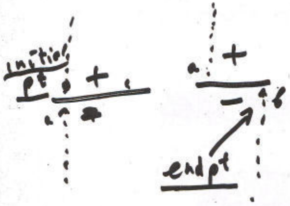
$$Q \in \mathbb{Z}, \quad Q = (1,2)$$

These elements are computed in our work.

Topological Types

Consider all possible 2-street passages (1, 2)

in the torus T_1^2 in the torus T_2^2



Segment S will measure m

Every passage has a form

$\langle \alpha \beta' \rangle = 1_\alpha \, 2_{\beta'}$, $\alpha = 1, 0, 2$
$\beta' = 1, 0, 2$

$\rho_{\alpha\beta'}$ - measure of this passage

I. ρ_{12}, ρ_{02}, ρ_{21}, ρ_{20}, ρ_{22}
$\sigma = 32541$

II. ρ_{12}, ρ_{01}, ρ_{02}, ρ_{21}, ρ_{20}
$\sigma = 24153$

III. ρ_{10}, ρ_{12}, ρ_{00}, ρ_{21}, ρ_{20} $\sigma = 41523$

IV. $\rho_{12}, \rho_{01}, \rho_{00}, \rho_{02}, \rho_{21}$
$$\sigma = 25314$$

V. $\rho_{10}, \rho_{21}, \rho_{01}, \rho_{00}, \rho_{21}$
$$\sigma = 31524$$

VI. $\rho_{11}, \rho_{10}, \rho_{12}, \rho_{01}, \rho_{02}$
$$\sigma = 52134$$

Every 2-street passage $P_{\alpha\beta'}$ represents an element in the group $\pi_1(\underline{\Pi}, s^+) = \pi_1(\underline{\Pi}, pt.)$ depending on the parameters

$$|a_1^\#|, |b_1^\#|, |a_2^\#|, |b_2^\#|, m = |S|$$

($a_j^!, b_j^!$ are exactly the specific canonical cycles in (T_j^2, s) calculated above.)

$a_j^!, b_j^!$ can be calculated through $(a_j, b_j, m = |S|)$.