HOLOMORPHIC CURVES AND RIEMANN-HILBERT PROBLEMS IN LOOP SPACES

G.KHIMSHIASHVILI

Abstract. Geometric and topological aspects of holomorphic curves in loop spaces are discussed with an emphasis on their relations to boundary value problems of Riemann-Hilbert type. A general formulation of Riemann-Hilbert problem in the setting of almost complex manifolds is presented and illustrated by considering some visual examples. In particular, analytic discs in a loop space of a three-dimensional Euclidean space are characterized in terms of a certain nonlinear partial differential equation. As an application, it is shown that the Hopf fibration can be considered as a solution to Riemann-Hilbert problem in the so-called Brylinski loop space of a standard three-dimensional sphere.

Key words: almost complex manifold, holomorphic curve, Riemann-Hilbert problem, Hopf fibration, Villarceau circle
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Introduction

A general concept of holomorphic curve in an almost complex manifold was suggested by M.Gromov [5]. As is well known, some loop spaces have natural almost complex structures (see, e.g., [3], [12], [14]), so one can define holomorphic curves in such loop spaces. Since holomorphic curves in complex vector spaces are closely related to boundary value problems for holomorphic functions it is natural to investigate those relations in the context of loop spaces. Our aim is to make first few steps in this direction.

In particular, in the first section we suggest a general setting for holomorphic curves and Riemann-Hilbert problems in the context of almost complex manifolds in the spirit of [5], [6], [18], [11]. The general scheme is then applied to holomorphic functions with values in the space of immersed loops in a three-dimensional Riemannian manifold (3-fold). For brevity, in this context we speak of *loopy holomorphic curves* and *loopy Riemann-Hilbert problems* .

It turns out that such functions and their images are naturally connected with several classical geometric and analytic concepts. In particular, a typical geometric picture of a loopy analytic disc is given by a solid torus foliated by closed curves governed by a certain partial differential equation. Thus one can consider various geometric problems concerned with foliated solid tori which are closely related to solving loopy Riemann-Hilbert problems .

In order to make the presentation self-contained (at least formally), in the second section we describe the basic construction of a natural almost complex structure on the space of immersed loops in a Riemannian 3-fold. This construction goes back to J.-L.Brylinski and L.Lempert, and we closely follow [3], [12].

In the third section we discuss geometric models of loopy analytic discs . Since the topic is rather subtle, our considerations are restricted to loops in a three-dimensional real affine space. In this case an explicit description of analytic discs is possible in terms of a certain partial differential equation suggested in [12] which we call the *Lempert equation*. A similar equation was obtained in [12] even in bigger generality, but the Euclidean setting accepted in this paper enables us to give a natural representation of solutions to Lempert equation in terms of conformal mappings (Theorem 2).

Our approach relies on the so-called *Weisbach theorem* called sometimes the *fundamental theorem of axonometry* [1], so we briefly recall it in a separate section and then explain its relation to Theorem 2. The Weisbach theorem in this context was brought to the author's attention by Elias Wegert. The author acknowledges inspiring discussions with E.Wegert in Bergakademie Freiberg which were crucial for realizing connections between the Lempert equation and conformal mappings. It may be added that J.Weisbach was one of the Rectors of Bergakademie in XIXth century and E.Wegert kindly provided a lot of interesting information about Weisbach theorem and related results.

In conclusion we present an interpretation of the Hopf fibration as a geometric picture of holomorphic curve in the loop space of 3-sphere (Theorem 3). This result can be expressed by saying that a round 3-sphere is a holomorphic sphere in the space of immersed loops on itself, which sounds curious and may hopefully serve as a pattern for a new topic concerned with representation of 3-folds as loopy holomorphic curves.

1. Riemann-Hilbert problems in almost complex setting

Nowadays there exist a number of commonly used concepts of Riemann-Hilbert problem . We only deal with Riemann-Hilbert problems considered as *boundary value problems for holomorphic functions*. A general formulation of Riemann-Hilbert problem of such type was given in [6]. Here we elaborate upon the definition from [6] so that it becomes applicable to holomorphic functions of one complex variable with values in loop spaces of certain types.

To begin with, we present a version of the general definition from [6] which is appropriate for considering holomorphic functions of one complex variable with values in loop spaces of certain types. Recall that an almost complex structure J on a smooth manifold M is defined as a smooth family of linear operators $J_p = J(p)$ in tangent spaces $T_pM, p \in M$, such that $J_p^2 = -I$ (here and in the sequel I always denotes the identity mapping of the corresponding space). In particular, each complex manifold (for example, $\mathbb{C}^n$ or $\mathbb{CP}^n$) has a canonical complex structure defined by the operator of multiplication by $\imath$ in each tangent space. The concept of holomorphic mapping between complex manifolds is generalized in the context of almost complex manifolds as follows.

Consider two almost complex manifolds (M, J) and (N, J'). A differentiable mapping $F : M \to N$ is called *holomorphic* if its differential dF intertwines the given almost complex structures, namely:

$$dF(p)J_p = J'_{F(p)}dF(p), \tag{1}$$

for each $p \in M$. Sometimes such mappings are called *pseudo-holomorphic* (cf. [5]) but we prefer to omit the prefix "pseudo" since this cannot lead to a misunderstanding in the sequel. As is well known, the local description of such mappings is closely related to *Bers-Vekua equation* and *generalized analytic functions* [5].

It is easy to verify that, for finite-dimensional complex manifolds, the above definition gives the usual concept of holomorphic mapping. In particular, taking a domain in the complex plane endowed with the canonical complex structure we get a concept of holomorphic function of one complex variable with values in an almost complex manifold N. If M or/and N are infinite-dimensional complex manifolds modelled on complex Banach spaces, proving equivalence of the two definitions of holomorphic map requires some care but we need not discuss here those nuances.

If M is a one-dimensional complex manifold (Riemann surface) then the image of a holomorphic mapping $M \to N$ is called a *holomorphic curve* in N. In particular, if M is a domain in $\mathbb{C}$, such an image is called an *analytic disc.* If $M = \mathbb{CP}$ is the Riemann sphere then its holomorphic images are called *holomorphic spheres.* Obviously, a holomorphic sphere is a union of two analytic discs glued along their boundaries. In the last section we present an example of such situation in the loop space of a 3-sphere.

In order to formulate Riemann-Hilbert problem in almost complex setting, suppose moreover that M is decomposed into two (open) parts M_+, M_- by a smooth divisor (hypersurface) Γ. Introduce the function spaces as follows. For an open subset $U \subset M$, let $A(U,N)$ denote the set of all mappings defined and continuous in $\overline{U}$ taking their values in N and holomorphic in U. Fix finally a continuous mapping (current) Φ on Γ with values in a subgroup G of infinite-dimensional Lie-Frechet group Diff N consisting of smooth diffeomorphisms of N.

Then Riemann-Hilbert problem defined by quintiple (M, N, Γ, G, Φ) is formulated as the problem of describing the totality of pairs $(X_+, X_-) \in A((M_+, N) \times A(X_-, N))$ satisfying the following condition on divisor Γ:

$$X_+(p) = \Phi(p)(X_-(p)),\ p \in \Gamma, \tag{2}$$

where $\Phi(p)$ acts on $X_-(p)$ as an element of Diff$N$. Notice that by taking $M = \mathbb{C}, \Gamma = \{|z| = 1\}, N = \mathbb{C}^n, G = GL(n, \mathbb{C})$, and some $(n \times n)$-matrix-function on Γ in the role of Φ, one obtains a classical version of Riemann-Hilbert problem called the *problem of linear conjugation* (cf. [13], [18], [7]).

Thus we see that each pair of almost complex manifolds (M, N) yields a collection of analytic problems whose nature strongly depends on the geometry and topology of the manifolds and group G considered. The classical theory of Riemann-Hilbert problem and singular integral equations appears as a particular case of this general scheme. In a short note like this one it makes no sense to discuss the general scheme in depth so we just briefly outline some new aspects and phenomena appearing when N is taken to be an infinite dimensional almost complex manifold.

It should be noted that several classes of examples of such manifolds N are already known. Some of them arise from loop spaces of various kinds, the most famous examples being given by a group of loops in a compact Lie group [14] and group Diff S^1/S^1 [14], [12]. Up to our knowledge, Riemann-Hilbert problems of the above type have never been discussed in the literature even for these concrete examples so we make an attempt to fill this gap.

It appears more convenient to use a little bit more geometric language. Namely, in complete analogy with the finite-dimensional setting, while dealing with Riemann-Hilbert problems in loop spaces one is inevitably led to considering images of holomorphic mappings of discs into a given loop space. Thus it becomes necessary to deal with analytic discs in certain loop spaces which could be called *loop-valued analytic discs* in order to distinguish from the usual analytic discs in finite-dimensional complex manifolds. For brevity, we call them *loopy analytic discs*. Similar geometric objects in loop spaces of 3-folds were earlier considered by L.Lempert [12] and appeared to be related to interesting geometric constructions.

Actually, in many cases solving Riemann-Hilbert problems can be reduced to constructing analytic discs with boundaries in a prescribed manifold (usually called a *target manifold* [18]). The latter problem is of a geometric flavour and appears more flexible from the viewpoint of various generalizations so we concentrate on constructing loopy analytic discs.

2. Immersed loops in Riemannian 3-folds

In this section, we explicate the above general considerations in the context of certain loop spaces of Riemannian 3-folds. Let us give the precise definition of the appropriate loop space together with a canonical almost complex structure on it. Given a Riemannian 3-fold M, the space we are going to deal with consists of the so-called *oriented immersed loops* in M [12]. Such loop spaces were carefully investigated by J.-L.Brylinski [3]. For this reason we call it the *Brylinski loop space* and denote by BM in order to distinguish from the usual space LM of loops considered with fixed parameterizations. We only deal with smooth (meaning C^∞) loops.

Let M be an oriented Riemannian 3-fold. Denote by BM the set of equivalence classes of smooth immersions $f : S^1 \to M$ with respect to a natural equivalence relation: two loops $f_1, f_2 : S^1 \to M$ are equivalent if $f_1 = f_2 \circ \varphi$, where φ is an orientation preserving diffeomorphism (reparameterization) of S^1. Elements of BM are called *immersed loops.* At each point $p \in M$ of an immersed loop $\gamma = [f]$ one has a well-defined tangent direction since we only permit orientable changes of variables. Let us denote the unit vector in this direction by t_p. The set BM is endowed by a natural topology using the sections of normal bundle.

Namely, fix an immersed loop $\gamma \in BM$ represented by $f : S^1 \to M$ and let N_f denote its normal bundle, i.e. the union of normal spaces to $\gamma = \operatorname{im} f$ in TM. This makes sense since M has Riemannian metric and one can, moreover, construct the (partially defined) exponential map $\exp : N_f \to M$. Bundle N_f inherits a Riemannian metric and connection from TM so it makes sense to speak of C^k-norms of its smooth sections. Hence one can introduce Whitney (or C^∞) topology on its smooth sections. The images of sufficiently small neighbourhoods of the origin under exp form by definition the basis of topology on BM. As was shown in [3], BM endowed with this topology becomes a Frechet manifold modelled on the space of smooth mappings from S^1 into $\mathbb{R}^2$. The tangent space $T_{[f]}BM$ can be naturally identified with $C^\infty(N_f)$ and its elements can be interpreted as normal vector fields along f.

This enables one to introduce an almost complex structure on BM in a pretty visual way. Namely, an endomorphism $J : T_{[f]}BM \to T_{[f]}BM$ is defined as follows. Let $v \in T_{[f]}BM$ and denote by $v(p)$ the corresponding normal vector at point

$p \in \mathrm{im}[f]$. Then we put $Jv = w$, where w is another normal vector field along f such that, for each point $p \in \mathrm{im} f$, vector $w(p)$ is orthogonal to $v(p)$ and $t(p)$, has the same length as $v(p)$, and the triple $(v(p), w(p), t(p))$ is positively oriented (with respect to a given orientation of M). This obviously provides a rigorous definition of J. Actually, its action on a normal vector field along $[f]$ consists in rotation by $\frac{\pi}{2}$ in positive direction when looking from the endpoint of tangent vector $t(p)$ (or equivalently the vector product $v(p) \times w(p)$ points in the direction of tangent vector $t(p)$). It is pretty obvious that $J^2 = -I$ so that J defines an almost complex structure on BM which is called the canonical almost complex structure.

We wish to study analytic discs in Brylinski loop space BM, where M is an oriented Riemannian 3-fold as above. An especially visual and practically important case is when the ambient manifold is a *three-dimensional real vector space with Euclidean metric* which we call *Euclidean 3-space* for short. Another visual case is when one takes the ambient manifold to be a three-dimensional sphere S^3 with the standard (round) metric induced from the ambient four-dimensional Euclidean space. Such a sphere will be called a *Euclidean 3-sphere* or *round 3-sphere*.

Our next aim is to obtain a more explicit description of loopy analytic discs in Euclidean 3-space and round 3-sphere. In fact, in both these cases it even appears possible to give a visual analytic interpretation for loopy analytic discs and relate them to certain nonlinear differential equations, which will be our main occupation in the rest of this paper.

3. Loopy analytic discs in $\mathbb{R}^3$

Let us recall the analytic description of Euclidean loopy analytic discs obtained by L.Lempert [12]. First we describe the typical geometric picture of a generic loopy analytic disc in any 3-fold M. Notice that geometrically it is just a 2-parametric family of loops in M with a parameter running on some disc which at the first step can be assumed sufficiently small. Then it is easy to imagine that generically loops in the family are non-singular and mutually disjoint. Thus their union should be homeomorphic to direct product $D^2 \times S^1$, i.e., to a solid torus $\mathbb{T}$. Moreover, this solid torus comes together with a foliation by closed curves in which case we speak of a *foliated solid torus*. This is actually a well-known pattern in three-dimensional topology because such objects serve as building blocks for the so-called *Seifert fibrations* [15], [10].

A natural problem which immediately arises in this context, is to investigate if a given foliated solid torus can be represented as the image of an analytic disc. One important case when this can be done was indicated by L.Lempert [12]. Namely, in the case of Euclidean 3-space, loopy analytic discs can be constructed from solutions to a rather simple nonlinear differential equation for functions of three real variables.

To this end L.Lempert associated a complex-valued function with a given foliated solid torus $\mathbb{T} \subset \mathbb{R}^3$ as follows. Intersect $\mathbb{T}$ with a hypersurface H which transversally meets each leaf in a single point. Then the intersection $H \cap \mathbb{T}$ is diffeomorphic to a disc $D \subset \mathbb{C}$ and can be identified with this disc, so its points can be simply considered as complex numbers. Notice that by our assumption, for each point $p = (x, y, z) \in \mathbb{T}$, there exists a unique leaf passing through p and by construction it intersects $H \cap \mathbb{T}$ in a single point which we denote $u(x, y, z)$. This correspondence obviously defines a function $u : \mathbb{T} \to \mathbb{C}$ and from general principles it follows that generically u is a differentiable complex-valued function on U. We call it an *generating function* of foliated solid torus $\mathbb{T}$. Obviously, it depends on the choice of transversal hypersurface H.

Among other results, L.Lempert proved in [12] that if a given foliated solid torus T arises as the image of a loopy analytic disc on Euclidean 3-space, then H can be chosen in such way that the generating function u satisfies the following nonlinear partial differential equation:

$$u_x^2 + u_y^2 + u_z^2 = 0, \tag{3}$$

which we call the *Lempert equation.* Actually, in this case the intersection $H \cap T$ can be obtained by taking values of loops at any fixed point $t \in S^1$.

The above equation is quite interesting by itself and its analysis is crucial for constructing loopy analytic discs. Actually, a sort of converse is true: speaking informally, loopy analytic discs can be constructed from solutions to the Lempert equation. This result is especially important for us, so we formulate it as a theorem. The proof is omitted because it is actually just a little bit more precise version of a similar statement established in [12].

Theorem 1. *([12]) If $\mathbb{T}$ is a foliated solid torus in Euclidean 3-space and there exists a hypersurface H such that the generating function u is differentiable and satisfies equation (3) then the leaves of foliation can be reparameterized in such a way that $\mathbb{T}$ becomes the image of a loopy analytic disc.*

Thus for constructing loopy analytic discs and solving loopy Riemann-Hilbert problems it is crucial to have at hand an ample set of solutions to Lempert equation. We proceed towards that aim by relating its solutions to conformal mappings in three dimensions. Our approach relies on an elementary but very useful (e.g., in technical drawing) geometric theorem proved by Julius Weisbach in 1844. In the next section we recall this theorem and explain how it can be applied to constructing loopy analytic discs.

4. Weisbach theorem and conformal lifts

Consider the three-dimensional Euclidean space $E = \mathbb{R}^3$ with canonical coordinates (x, y, t) and identify the hyperplane $L = \{t = 0\}$ with the standard complex plane $\mathbb{C}$ by putting $z = x + \imath y$. Under an *orthogonal frame* in E we understand any triple of radius-vectors (i.e., vectors with the initial point at the origin) which are mutually orthogonal and have the same length. Recall that three points of L are called *contactical* if they can be represented as the images of orthogonal projection of the ends of some orthogonal frame. The following classical result is easy to prove but quite useful (see, e.g., [1]).

Weisbach theorem. *Three points $p, q, r \in L \cong \mathbb{C}$ are contactical if and only if*

$$p^2 + q^2 + r^2 = 0. \tag{4}$$

This result permits a visual geometric interpretation which makes its proof nearly evident (notice the "only if" part is verified by an elementary computation). In order to proof the "if" part, consider the triples of real and complex parts of p, q, r as vectors X, Y in $\mathbb{R}^3$. Then it is immediate that condition (4) implies that X and Y are orthogonal and have the same length $d = |X| = |Y|$. It is also easy to see that in this setting our task is just to find a third vector Z such that it is orthogonal to both X and Y and has the same length. Existence of such vector is geometrically obvious but one can also write a useful explicit formula in terms of vector product, namely:

$$Z = \pm \frac{X \times Y}{d}, \tag{5}$$

where the sign can be chosen according to a desired orientation of the triple (X, Y, Z). This completes the proof and reveals the geometric meaning of Weisbach theorem.

Moreover, such an interpretation of this result enables one to establish a very explicit relation between conformal mappings and solutions of Lempert equation. Indeed, let us consider a germ of conformal mapping $\Phi : E \to E$. Then by conformality condition its derivative (differential) transforms each orthogonal frame into an orthogonal one. Consider now the composition $v = P \circ \Phi$, where P is the orthogonal projection on L. Since P is linear, using the chain rule on easily derives that v is a solution to Lempert equation. Thus generating functions for Euclidean loopy analytic discs can be obtained from germs of conformal mappings. This is the key result in the sequel and so we formulate it as a theorem.

Theorem 2. *Each germ h of conformal map in $\mathbb{R}^3$ defines a germ of solution to Lempert equation (3) by taking its composition $P \circ h$ with orthogonal projection P.*

Taking into account Theorem 1, this result provides an ample source of loopy analytic discs . One may wonder about the converse, namely, if each germ u of solution to Lempert equation (3) can be obtained in such way. If this is the case, i.e., if there exists a germ of conformal mapping h such that $u = P \circ h$ then any such germ is called a *conformal lift* of u. Since by Liouville theorem the storage of conformal maps in a three-dimensional Euclidean space is rather poor, there is no hope that conformal lifts always exist. Therefore it is desirable to to indicate some reasonable criterion or sufficient conditions for the existence of conformal lifts. This problem can be successfully treated using Weisbach theorem and our comments to its proof.

More precisely, let u be a solution to Lempert equation (3) in domain U. Separating the real and imaginary part of its gradient we get two gradient vector fields X, Y in U. It is straightforward to verify that in order to obtain a conformal lift of u it is sufficient to find a function germ ψ such that its gradient $Z = \psi$ taken together with X, Y forms an orthogonal frame at each point $p \in U$. Notice that at each point the sought field Z can be expressed by the formula (5) with the sign chosen once and forever.

So the problem is reduced to showing that the vector field defined by this formula is actually a **gradient** vector field. To this end one can use the well known criterion in terms of vanishing of the curl (rotor). Using standard formulae of vector calculus one can now get an explicit expression for the curl of vector field defined by the formula (4.2) and write the conditions which guarantee vanishing of the rotor in question.

This gives a system of explicit partial differential equations which should be satisfied by u in order that it has a conformal lift. By an easy qualitative analysis of those equations one becomes convinced that conformal lifts do not always exist. Thus one should look for further methods of constructing solutions to Lempert equation, which is an interesting open problem. However, the above results already enable one to construct many examples of loopy analytic discs in $\mathbb{R}^3$. Some of them can be written explicitly so that one becomes able to visualize loopy analytic discs in a quite satisfactory way.

Another merit of the above connection between loopy analytic discs and conformal mappings is its conformal invariance so applying the stereographic projection $S^3 \to \mathbb{R}^3$ one can obtain an analog of Lempert equation for a round 3-sphere and construct examples of loopy analytic discs using Theorem 1. As an illustration, in conclusion we describe a curious visual example of such kind arising from the Hopf fibration.

5. Hopf fibration as a holomorphic curve

For our purposes it is appropriate to define the Hopf fibration in complex setting. Consider the unit sphere $S^3 \subset \mathbb{C}^2 \cong \mathbb{R}^4$ and the Riemann sphere $\mathbb{P} = \overline{\mathbb{C}} \cong S^2$. Then the Hopf fibration $H : S^3 \to S^2$ is defined by sending each point $(z_1, z_2) \in S^3$ into the ratio of its coordinates interpreted as a point of $\mathbb{P}$, i.e. $H(z_1, z_2) = z_1/z_2$. It is evident that fibres of H are the the *complex great circles*, i.e. intersections of complex lines in $\mathbb{C}^2$ with S^3, so one can consider its "inverse" as a map from S^2 into the space of smooth loops on S^3.

Let us endow S^3 with the standard Riemannian metric inherited from the ambient Euclidean space. The sphere S^3 endowed with this metric will be called the round 3-sphere and denoted S^3_r. We can now consider the corresponding Brylinski loop space BS^3_r and get a map $H^{-1} : \mathbb{P} \to BS^3_r$. Thus it becomes possible to treat the latter map from the viewpoint developed in previous sections. A straightforward calculation shows that its differential dH^{-1} intertwines the almost complex structures on $\mathbb{P}$ and BS^3_r and so it defines a holomorphic curve in BS^3_r. We omit this calculation because the same conclusion may be obtained in a much more direct and visual way by considering the examples of analytic discs in $B\mathbb{R}^3$ described below.

Correspondingly, the restriction of H^{-1} on any disc in $\mathbb{P}$ defines a loopy analytic disc in BS^3_r. In particular, taking the unit disc and its complement we get a solution to loopy Riemann-Hilbert problem (2) with the constant coefficient whose value at each point $p \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is the identity mapping of BS^3_r. In terms of analytic discs, one can state that S^3_r is the union of images of two loopy analytic discs. Despite their simplicity, these observations seem instructive so we formulate them as a theorem.

Theorem 3. *The map H^{-1} defines a holomorphic embedding of the Riemann sphere into the space of oriented immersed loops on S^3_r, in other words, H^{-1} defines a holomorphic curve in BS^3_r. In particular, a round 3-sphere S^3_r can be represented as a union of two loopy analytic discs glued along their boundaries, i.e., S^3_r foliated by the complex great circles is a solution to a loopy Riemann-Hilbert problem in BS^3.*

Using the above observations and stereographic projection $\Pi : S^3 \to \mathbb{R}^3$ one can get a nice geometric picture in $\mathbb{R}^3$, which, in particular, enables one to visualize certain analytic discs in $\mathbb{R}^3$. It is well known that the image of the unit disc under $\Phi = \Pi \circ H^{-1}$ is a solid torus $\mathbb{T}$ bounded by a round torus (torus of revolution) $\mathbb{T} \cong T^2$ in $\mathbb{R}^3$ (see, e.g., Ch.10 in [1]). The same holds for any disc in $\mathbb{C} \subset \mathbb{P}$ centered at the origin.

It is also known (but probably not so "well-known") that the images of complex great circles under Π are genuine (metric) circles which have been discovered by

I.Villarceau [17] (nowadays they are called *Villarceau circles* [1]). They can be defined as the intersections of a round torus T_r^2 with the bitangent plane passing through the center of torus T_r^2 [1]. Thus the preimages $\Phi^{-1}(w)$ of points w from the unit disc are exactly the Villarceau circles.

On each round torus T_r^2, Villarceau circles come in two families each of which consists of nonintersecting circles. Two Villarceau circles belonging to the same family will be called *coherent*. Thus each of the two families of coherent Villarceau circles defines a foliation of a round torus. Any two circles in the same family on a given round torus are linked with the linking number 1 which corresponds to the well-known fact that the Hopf invariant of Hopf fibration is equal to one.

Consider now a *round solid torus* $\mathbb{T}_r$ defined as the closure of interior of a round torus T_r^2. Obviously, $\mathbb{T}_r$ is a union of continual family of coaxial round tori lying inside $\mathbb{T}_r$ and the axial circle which is equal to the intersection of their interiors. One sees now that the family of coherent Villarceau circles of all those round tori can be chosen in such a way that together with axial circle they give a foliation of $\mathbb{T}_r$ by loops (circles) which are mutually linked with the linking number 1. Let us call this picture a *Villarceau solid torus*.

Comparing these observations with the discussion in Section 3 we see that a Villarceau solid torus geometrically resembles the geometric picture of analytic disc described in the preceding section. In fact, Theorem 1 can be used to show that Villarceau foliations really define loopy analytic discs in $\mathbb{R}^3$.

To this end we simply indicate a generating function in the sense of Section 3 which is naturally associated with the above geometric picture. Consider the following complex-valued function of three complex variables

$$u(x,y,z) = \frac{x^2+y^2+(z-\imath)^2}{x+\imath y}$$

which is merely the composition $\Pi_2 \circ H \circ \Pi^{-1}$ written in Cartesian coordinates, where Π_2 denotes the stereographic projection $S^2 \to \mathbb{R}^2 \cong \mathbb{C}$.

Taking into account the above remarks we conclude that the preimage of each point $w \in \mathbb{C}$ is a Villarceau circle. Moreover, the full preimage of any circle centered at the origin is a round solid torus. Now it is easy to see that the function u is a generating function for Villarceau solid tori described above. In order to see that it really makes each of those solid tori into a loopy analytic discs it remains to verify that it satisfies the Lempert equation, which is a matter of elementary check. By Theorem 2 this means that each of those Villarceau round tori gives a precise picture of a loopy analytic disc in $\mathbb{R}^3$ which we call *Villarceau toroid*. Thus we have established the following final result.

Proposition 1. *Each round solid torus in $\mathbb{R}^3$ foliated by Villarceau circles is the image of an analytic disc in $B\mathbb{R}^3$.*

To our mind, this beautiful geometric picture alone gives a sufficient justification for the setting and considerations presented above. We wish now to make a few remarks related to the latter proposition. First of all, since the loopy analytic discs in S_r^3 defined by H^{-1} and Villarceau toroids are related by the stereographic projection which is a conformal map, from the chain rule and above computation for function u it follows that H^{-1} is indeed a holomorphic map into BS_r^3. This gives a simple proof of Theorem 3 avoiding explicit use of the round metric on S_r^3. It is also worthy of noting that Proposition 1 could be derived from Theorem 2 by showing that the function u has a conformal lift, which follows from its relation to stereographic mapping. In particular, one can verify the vanishing of the rotor of the normalized vector product from the formula (5).

Using methods of nonlinear analysis it is possible to show that one can deform the Villarceau toroid in such a way that all leaves of the foliation remain closed and the generating function is again a solution to Lempert equation. Such deformations can be described by explicit equations using methods of deformation theory. For us the main point is that they provide examples of loopy analytic discs different from Villarceau toroids.

Proposition 2. *There exist small perturbations of the Villarceau toroid which can be represented as the images of loopy analytic discs.*

This fact may be used to construct solutions to loopy Riemann-Hilbert problems with non-constant coefficients which are sufficiently close to the identity. It would be very interesting to construct similar examples with coefficients not necessarily close to identity. Clearly, similar constructions and results make sense for other 3-folds foliated by loops, for example, for tangent circle bundles of compact orientable two-dimensional surfaces without boundary. One can also consider similar problems for Seifert fibrations [15], [10].

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A.Razmadze Mathematical Institute
M.Aleksidze str.1, Tbilisi 0193, Georgia
E-address: khimsh@rmi.acnet.ge