I. Introduction

I.1 Bi-Hamiltonian Formulation

$$q_t = q_{xxx} + 6qq_x, \quad -\infty < x < \infty, \quad t > 0.$$

$$\begin{cases} q_t = \partial_x[q_{xx} + 3q^2] \\ \\ q_t = (\partial_x^3 + 2q\partial_x + 2\partial_x q)q. \end{cases}$$

$$\begin{cases} q_t = \partial_x \frac{\delta}{\delta q} \int_{-\infty}^{\infty} \left(-\frac{1}{2}q_x^2 + q^3\right) dx \\ \\ q_t = \left(\partial_x^3 + 2q\partial_x + 2\partial_x q\right) \frac{\delta}{\delta q} \int_{-\infty}^{\infty} \frac{1}{2}q^2 dx. \end{cases}$$

$$L \doteqdot \partial_x^{-1} \left[\partial_x^3 + 2q\partial_x + 2\partial_x q\right]$$

1+1: MAGRI (1978), Gelfand-Dorfman (1980), F-Fuchssteiner (1980)

2+1: F with Dorfman, Gelfand, Santini (1988)

Integrable Nonlocal Nonlinear Equations in $n+1$ (2004).

I.2 Lax pair formulation

GGKM 1967, LAX 1968, Zakharov and Shabat 1972.

$$iq_t + q_{xx} - 2|q|^2 q = 0$$

This PDE is the compatibility condition of the following two ODEs satisfied by the 2×2 matrix $\Psi(x, t, k)$:

$$\Psi_x + ik\sigma_3\Psi = Q\Psi, \quad k \in \mathbb{C},$$

$$\Psi_t + 2ik^2\sigma_3\Psi = (2kQ - iQ_x\sigma_3 - i|q|^2\sigma_3)\Psi,$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ \bar{q}(x,t) & 0 \end{pmatrix}.$$

1. Advanced Form of Separation of Variables.

2. Nonlinear Fourier Transform.

II. Analytical Inversion of Integrals-Imaging

Ablowitz-F, Beals and Coifman (1982)

$$\frac{\partial \Psi}{\partial x_1} + i\sigma_3 \frac{\partial \Psi}{\partial x_2} + k\sigma_3 \Psi = Q\Psi, \quad k \in \mathbb{C}$$

Nonlinear FT in 2D via the $\bar{\partial}$ formulation

F-GELFAND (1992)

$$(\partial_{x_1} + i\partial_{x_2} - k)\,\mu(x_1, x_2, k) = q(x_1, x_2)$$

Novel derivation of 2D FT via $\bar{\partial}$

F-NOVIKOV (1992)

$$\left[\frac{1}{2}\left(k + \frac{1}{k}\right)\partial_{x_1} + \frac{1}{2\mathrm{i}}\left(k - \frac{1}{k}\right)\partial_{x_2}\right]\mu(x_1, x_2, k) = f(x_1, x_2)$$

Novel derivation of Radon transform

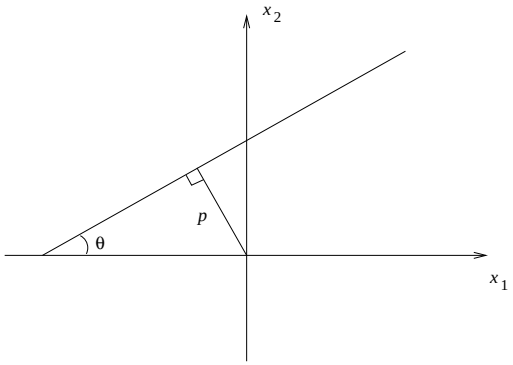
NOVIKOV (2003), F (2004)

$$\left[\frac{1}{2}\left(k + \frac{1}{k}\right)\partial_{x_1} + \frac{1}{2\mathrm{i}}\left(k - \frac{1}{k}\right)\partial_{x_2}\right]\mu(x_1, x_2, k) + f(x_1, x_2)\mu(x_1, x_2, k) = g(x_1, x_2)$$

Derivation of attenuated Radon transform

II.1 RADON TRANSFORM

Reconstruct f from its line integrals.



$$\begin{array}{l} x_1 = \tau\cos\theta - \rho\sin\theta \\ x_2 = \tau\sin\theta + \rho\cos\theta \end{array} \leftrightarrow \begin{array}{l} \tau = x_1\cos\theta + x_2\sin\theta \\ \rho = -x_1\sin\theta + x_2\cos\theta \end{array}$$

$$F(\tau,\rho,\theta) = f(\tau\cos\theta - \rho\sin\theta, \tau\sin\theta + \rho\cos\theta).$$

Direct Radon transform

$$\widehat{f}(\rho,\theta) = \int_{-\infty}^{\infty} F(\tau,\rho,\theta)\mathrm{d}\tau.$$

Inverse Radon transform

$$f(x_1,x_2) = \frac{1}{4\pi}(\partial_{x_1} - i\partial_{x_2})\int_0^{2\pi} e^{i\theta} G(\theta, -x_1\sin\theta + x_2\cos\theta)\mathrm{d}\theta,$$

$$G(\theta,\rho) = \frac{1}{i\pi}\oint_{-\infty}^{\infty} \frac{\widehat{f}(\rho',\theta)}{\rho' - \rho}\mathrm{d}\rho'.$$

II.2 MATHEMATICS OF SPECT

$$I = \int_L e^{-\int_{L(x)} f ds} g d\tau.$$

Reconstruct g from its weighted line integrals

Direct Attenuated Radon transform

$$\widehat{g}_f(\rho,\theta) = \int_{-\infty}^{\infty} e^{-\int_\tau^\infty F(s,\rho,\theta)ds} G(\tau,\rho,\theta)d\tau$$

Inverse Attenuated Radon transform

$$g(x_1,x_2) = \frac{1}{4\pi}(\partial_{x_1} - i\partial_{x_2})\cdot$$
$$\cdot\int_0^{2\pi} e^{i\theta} G(\theta, x_1\cos\theta + x_2\sin\theta, x_2\cos\theta - x_1\sin\theta)d\theta$$

where

$$G(\theta,\tau,\rho) = e^{\int_\tau^\infty F(s,\rho,\theta)ds}\Big\{e^{P^-\widehat{f}(\rho,\theta)}P^- e^{-P^-\widehat{f}(\rho,\theta)} + $$
$$+ e^{-P^+\widehat{f}(\rho,\theta)}P^+ e^{P^+\widehat{f}(\rho,\theta)}\Big\}\,\widehat{g}_f(\rho,\theta)$$

and $P^\pm$ are the usual projectors

$$P^\pm g(\rho) = \pm\frac{g(\rho)}{2} + \frac{1}{2i\pi}\oint_{-\infty}^{\infty}\frac{g(\rho')}{\rho'-\rho}d\rho'.$$

Spectral analysis $\rightarrow$ Analytic inversion of integrals

$$\left\{\frac{1}{2}\left(k+\frac{1}{k}\right)\partial_{x_1}+\frac{1}{2i}\left(k-\frac{1}{k}\right)\partial_{x_2}\right\}\mu(x_1,x_2,k)=f(x_1,x_2)$$

$$k\in\mathbb{C},\quad (x_1,x_2)\in\mathbb{R}^2,\quad f(x)\in S(\mathbb{R}^2)$$

$$z\doteqdot\frac{1}{2i}\left(k-\frac{1}{k}\right)x_1-\frac{1}{2}\left(k+\frac{1}{k}\right)x_2$$
$$\bar{z}=-\frac{1}{2i}\left(\bar{k}-\frac{1}{\bar{k}}\right)x_1-\frac{1}{2}\left(\bar{k}+\frac{1}{\bar{k}}\right)x_2$$

$$\frac{1}{2i}\left(\frac{1}{|k|^2}-|k|^2\right)\frac{\partial\mu}{\partial\bar{z}}=f,\quad |k|\neq 0$$

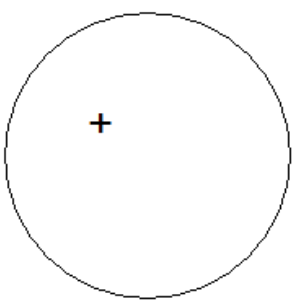
We look for μ such that $\mu=O\left(\frac{1}{z}\right),\quad z\rightarrow\infty.$

$$\boxed{\mu(x_1,x_2,k)=\frac{1}{2\pi i}\mathrm{sgn}\left(\frac{1}{|k|^2}-|k|^2\right)\iint_{\mathbb{R}^2}\frac{f(x_1',x_2')dx_1'dx_2'}{z'-z},\ |k|\neq 1}$$

Recall: $\dfrac{\partial\mu}{\partial\bar{z}}=f,\quad z\in D$, then

$$\mu=\frac{1}{2\pi i}\iint_D\frac{f(\zeta,\bar{\zeta})d\zeta\wedge d\bar{\zeta}}{\zeta-z}+\frac{1}{2\pi i}\int_{\partial D}\frac{\mu(\zeta,\bar{\zeta})}{\zeta-z}d\zeta$$

$$\mu = \frac{1}{2\pi i}\int \frac{(\mu^+ - \mu^-)(k')dk'}{k' - k},\ k' = e^{i\theta'},\ |k| \neq 1.$$



Complex k–plane

$$\mu^+ = -P^-\widehat{f}(\rho,\theta) - \int_\tau^\infty F(\rho,s,\theta)ds$$

$$\mu^- = P^+\widehat{f}(\rho,\theta) - \int_\tau^\infty F(\rho,s,\theta)ds$$

$$\mu^+ - \mu^- = -\frac{1}{i\pi}\oint \frac{\widehat{f}(\rho',\theta)d\rho'}{\rho' - \rho}$$

$$\boxed{\mu(x_1,x_2,k) = -\frac{1}{2i\pi^2}\int_0^{2\pi} \frac{e^{i\theta'}}{e^{i\theta'}-k}\left(\oint_{-\infty}^{\infty} \frac{\widehat{f}(\rho',\theta')\mathrm{d}\rho'}{\rho' - (x_2\cos\theta' - x_1\sin\theta')}\right)\mathrm{d}\theta'}$$

III. BVPs - D to N Maps

$$q_t + i\omega(-i\partial_x)q = 0,$$

$\omega(k)$: real polynomial of degree n

$$iq_t + q_{xx} = 0, \quad (x,t) \in D$$

$$\left[e^{-ikx+ik^2t}q\right]_t - \left[e^{-ikx+ik^2t}(iq_x - kq)\right]_x = 0, \quad k \in \mathbb{C}.$$

First consequence: A characterisation of the D to N map

$$\int_{\partial D} e^{-ikx+ik^2t}\,[qdx + (iq_x - kq)dt] = 0, \quad k \in \mathbb{C}.$$

Example 1.

$$D : 0 < x < \infty, \quad 0 < t < T$$

$$i \int_0^T e^{ik^2 s} q_x(0,s) ds - k \int_0^T e^{ik^2 s} q(0,s) ds,$$

$$= - \int_0^\infty e^{-ikx} q_0(x) dx + e^{ik^2 T} \int_0^\infty e^{-ikx} q(x,T) dx,$$

$Imk \leq 0.$

To solve for $q_x(0,x)$ use:

$$f(t) = \frac{1}{\pi} \int_{\partial\Omega^-} k e^{-ik^2 t} \left(\int_0^T e^{ik^2 s} f(s) ds \right) dk$$

Example 2

$D : l(t) < x < \infty, \quad 0 < t < T, \quad l''(t) > 0, \quad smooth.$

$$i \int_0^T e^{ik^2 s - ikl(s)} q_x(l(s), s) ds$$

$$= ... e^{ik^2 T} \int_{l(T)}^{\infty} e^{-ikx} q(x, T) dx, \quad Imk \leq 0.$$

Now use:

$$F(k) = \int_0^T e^{ik^2 s - ikl(s)} (-k - l'(s)) f(s) ds,$$

$$f(t) = -\frac{1}{\pi} \int_{\partial\Omega^-(t)} e^{-ik^2 t + ikl(t)} F(k) dk$$

$$+ \frac{1}{2\pi} \int_0^t K(s, t) f(s) ds$$

$$\mu_t + \left(ik^2 - ikl'(t)\right)\mu = \left(k - l'(t)\right)f(t)$$

Second consequence: Novel integral representations and an alternative characterisation of the D to N map

$$\left[e^{-ikx+ik^2t}q\right]_t - \left[e^{-ikx+ik^2t}(iq_x - kq)\right]_x = 0, \quad k \in \mathbb{C}.$$

$$\begin{cases} e^{-ikx+ik^2t}q = \left(e^{-ikx+ik^2t}\mu\right)_x \\ \\ e^{-ikx+ik^2t}(iq_x - kq) = \left(e^{-ikx+ik^2t}\mu\right)_t. \end{cases}$$

$$\begin{cases} \mu_x - ik\mu = q \\ \\ \mu_t + ik^2\mu = iq_x - kq \end{cases}$$

Example 1 $0 < x < \infty, \quad 0 < t < T.$

DIRECT

$$q_0(x) \to \widehat{q}_0(k) = \int_0^\infty e^{-ikx} q_0(x) dx, \qquad Imk \leq 0$$

$$\{q(0,t), q_x(0,t)\} \to$$
$$\widetilde{g}(k) = -k \underbrace{\int_0^T e^{ik^2 s} q_0(s) ds}_{\widetilde{g}_0(k^2 s)} + i \underbrace{\int_0^T e^{ik^2 s} q_x(0,s) ds}_{\widetilde{g}_1(k^2 s)}$$

INVERSE

$$\left\{ e^{-ik^2 t} \widehat{q}_0(k), \qquad e^{-ik^2 t} \widetilde{g}(k) \right\} \to q(x,t)$$

$$q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - ik^2 t} \widehat{q}_0(k) dk$$

$$+ \frac{1}{2\pi} \int_{\partial\Omega^+} e^{ikx - ik^2 t} \widetilde{g}(k) dk$$

Algebraic description of the D to N map

$$i\tilde{g}_1(k) - k\tilde{g}_0(k) = -\hat{q}_0(k) + e^{ik^2T}\hat{q}_T(k), \quad Imk \leq 0$$

$$i\tilde{g}_1(k) = -k\tilde{g}_0(k) - \hat{q}_0(-k) + e^{ik^2T}\hat{q}_T(-k), \quad \text{Im}k \geq 0$$

$$\tilde{g}(k) = -2k\tilde{g}_0(k) - \hat{q}_0(-k)$$

Sine-transform vs Ehrenpreis representation

$$q(x,t) = \int_L e^{ikx - ik^2t} d\rho(k)$$

Theorem. (F and Sung) Let $q(x,t)$ satisfy

$$\begin{aligned} q_t + q_x + q_{xxx} &= 0, \quad 0 < x < \infty, \quad 0 < t < T, \\ q(x,0) &= q_0(x) \in H^1(R^+) \\ q(0,t) &= g_0(t) \in H^1(0,T), \quad g_0(0) = q_0(0). \end{aligned}$$

Then

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+ik^3t}\widehat{q}_0(k)dk$$

$$+\frac{1}{2\pi}\int_{\partial\Omega^+} e^{ikx+ik^3t}[(1-3k^2)\tilde{g}_0(k)$$

$$+\frac{\nu_1-k}{\nu_2-\nu_1}\widehat{q}_0(\nu_2) + \frac{\nu_2-k}{\nu_1-\nu_2}\widehat{q}_0(\nu_1)]dk$$

ν_1, ν_2 : $\quad \nu - \nu^3 = k - k^3, \quad \nu^2 + \nu k + k^2 = 1$

$\Omega^+ = \{k \in \mathbb{C} : \operatorname{Im}\omega(k) > 0, \operatorname{Im} k > 0\}$

IBV PROBLEMS FOR INTEGRABLE NONLINEAR PDEs

Direct: Boundary Values $\rightarrow$ Spectral Functions

$$q_0(x) \rightarrow \widehat{Q}_0(k) = \begin{pmatrix} \bar{a}(k) & b(k) \\ \bar{b}(k) & a(k) \end{pmatrix}$$

via $\Psi_0(x,k)$, a solution of the x-part of the Lax pair evaluated at $t = 0$.

$$\{\underbrace{g_0(t)}_{q(0,t)}, \underbrace{g_1(t)}_{q_x(0,t)}\} \rightarrow \tilde{G}(k) = \begin{pmatrix} \bar{A}(\bar{k}) & B(k) \\ \bar{B}(\bar{k}) & A(k) \end{pmatrix}$$

via $\Phi_0(t,k)$, a solution of the t-part of the Lax pair evaluated at $x = 0$.

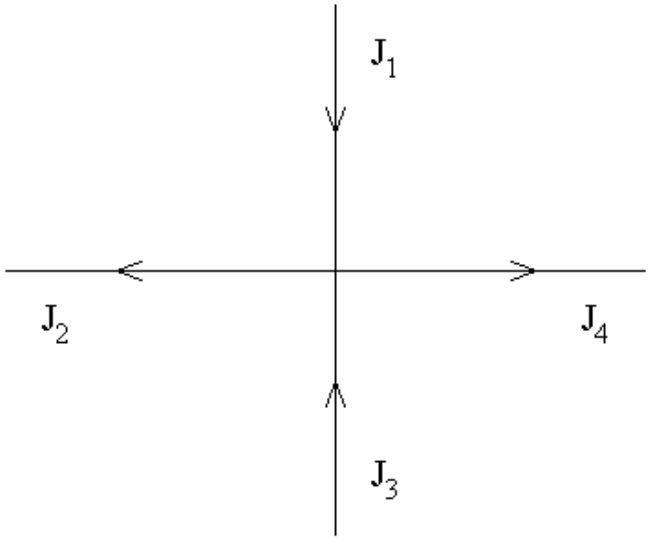
Inverse: $\{a(k), b(k)e^{-4ik^2t}, A(k), B(k)e^{-4ik^2t}\} \rightarrow q(x,t)$ via a 2×2 matrix RH problem.

$$q(x,t) = -\frac{1}{\pi}\left\{\int_{\partial\Omega^+} \Gamma(k)e^{-2ikx-4ik^2t}M_{11}\mathrm{d}k \right.$$
$$\left. + \int_{-\infty}^{\infty} \gamma(k)e^{-2ikx-4ik^2t}M_{11}\mathrm{d}k + \int_0^{\infty} |\gamma(k)|^2 M_{12}\mathrm{d}k \right\}$$

$$\boxed{\gamma(k) = \frac{b(k)}{\overline{a}(k)}}, \quad \boxed{\Gamma(k) = \frac{1}{a(k)\left(\frac{\bar{A}(\bar{k})}{\bar{B}(\bar{k})}a(k) - b(k)\right)}}$$

The 2×2 matrix-valued function $M(x,t,k)$ satisfies the following RH problem:

- M holomorphic in $k \in \mathbb{C}\backslash L$
- $M = I + O(\frac{1}{k})$, $k \to \infty$
- $M_- = M_+ J$, $k \in L$



$$J_1 = \begin{pmatrix} 1 & 0 \\ \Gamma(k)e^{2ikx+4ik^2t} & 1 \end{pmatrix},\ J_2 = J_3 J_4^{-1} J_1,$$

$$J_3 = \begin{pmatrix} 1 & \bar{\Gamma}(\bar{k})e^{-2ikx-4ik^2t} \\ 0 & 1 \end{pmatrix},$$

$$J_4 = \begin{pmatrix} 1 & -\gamma(k)e^{-2ikx-4ik^2t} \\ \bar{\gamma}(k)e^{2ikx+4ik^2t} & 1-|\gamma|^2 \end{pmatrix}.$$

The Generalised D to N map

The spectral functions satisfy the global relation

$$B(k) - R(k)A(k) = e^{4ik^2T}\frac{c(k)}{a(k)}, \quad k \in \Omega^+,$$

where $R(k) = \frac{b(k)}{a(k)}$, $\Omega^+ = \{k \in \mathbb{C}, \arg k \in [0, \pi/2]$ and $c(k)$ as a function of k is analytic for $k \in \Omega^+$, and is of $O(1/k)$ as $k \to \infty$.

1. Algebraic $q_x(0,t) - cq(0,t) = 0$.

$$\frac{B(k)}{A(k)} = -\frac{2k + ic}{2k - ic}\frac{b(-k)}{a(-k)}$$

2. The spectral functions are defined by

$$A(k) = e^{2ik^2T}\bar{\Phi}_2(T,\bar{k}), \quad B(k) = -e^{2ik^2T}\Phi_1(T,\bar{k}),$$

where the vector $\Phi(t,k)$ with components $\Phi_1(t,k)$ and $\Phi_2(t,k)$ satisfies the t-part of the Lax pair evaluated at $x = 0$:

$$\Phi_t + 2ik^2\sigma_3\Phi = (2kG_0(t) + G_1(t))\,\Phi, \quad t > 0, k \in \mathbb{C},$$

$$\Phi(0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where

$$G_0(t) = \begin{pmatrix} 0 & g_0(t) \\ \bar{g}_0(t) & 0 \end{pmatrix}, G_1(t) = \begin{pmatrix} -|g_0(t)|^2 & g_1(t) \\ -\bar{g}_1(t) & |g_0(t)|^2 \end{pmatrix}.$$

If $q_0(x) = 0$, then
$\int_0^T e^{4ik^2 s}[i|g_0(s)|^2\Phi_1(s,k)$

$-(2kg_0(s) + ig_1(s))\Phi_2(s,k)]ds = e^{4ik^2T}c(k).$
The global relation can be solved in CLOSED FORM!

$$g_1(t) = \frac{g_0(t)}{\pi}\int_{\partial\Omega^+} e^{-2ik^2t}(\Phi_2(t,k)-\Phi_2(t,-k))dk$$

$$+\frac{4i}{\pi}\int_{\partial\Omega^+} e^{-2ik^2t}kR(k)\bar{\Phi}_2(t,\bar{k})dk$$

$$+\frac{2i}{\pi}\int_{\partial\Omega^+}\Big(e^{-2ik^2t}k[\Phi_1(t,k)-\Phi_1(t,-k)] + ig_0(t)\Big)dk,$$

where $\partial\Omega^+$ is the oriented boundary of Ω^+ and if $a(k)$ has zeros in Ω^+ it must be deformed over these zeros. The definition of Φ implies
$|\Phi_1(t,k)|^2 + |\Phi_2(t,k)|^2 = 1, \quad Imk^2 = 0.$
This relation, together with the fact that the nonlinear ODEs satisfied by Φ_1 and Φ_2 are equivalent to nonlinear Volterra integral equations, imply global existence.

Asymptotics: Boutet de Monvel, Its, Kamvissis, Kotlyarov, Schultz

Nonlinear: Pelloni, Shepelsky, Sung, Treharne

Water Waves: Ablowitz, Musslimani

Linear Polygonal: Antipov, ben-Abraham, Charalabopoulos, Crowdy, Dassios, Dimakos, Kapaev, Spence

Linear Arbitrary: Delillo, Pinotsis, Shargorodsky, Zhou, Zyskin

Numerics: Flyer, Fonberg, Fulton, Iserles, Marinakis, Papatheodorou, Saridakis, Sifalakis, Xenophontos

$$f(z) = \frac{1}{2\pi^2}\int_{\partial D} \frac{(\zeta - z)^{-1}}{|\zeta - z|^2} J f(\zeta) - \frac{1}{2\pi^2}\int_D \frac{(\zeta - z)^{-1}}{|\zeta - z|^2}\overline{\partial_L} f(\zeta) d\zeta,$$

$$z \in D.$$

$$z = x_0 + x_i e_i, \quad \zeta = \xi_0 + \xi_i e_i$$

$$d\zeta = d\xi_0 \wedge d\xi_1 \wedge d\xi_2 \wedge d\xi_3$$

$$J = d\xi_1 \wedge d\xi_2 \wedge d\xi_3 - e_1 d\xi_0 \wedge d\xi_2 \wedge d\xi_3$$

$$-e_2 d\xi_0 \wedge d\xi_3 \wedge d\xi_1 - e_3 d\xi_0 \wedge d\xi_1 \wedge d\xi_2$$

$$\overline{\partial_L} f = \frac{\partial f}{\partial x_0} + e_j \frac{\partial f}{\partial x_j}$$

$$f(z) = \frac{1}{2\pi^2}\int_{\partial D} \left\{ \frac{(\zeta - z)^{-1}}{|\zeta - z|^2} J f + \frac{\frac{1}{2}}{|\zeta - z|^2}\overline{J}(\overline{\partial_L} f) \right.$$

$$\left. + \frac{-\frac{1}{4}}{\zeta - z} J \triangle f \right\} + \frac{1}{8\pi^2}\int_D \frac{1}{\zeta - z}\left(\overline{\partial_L}\triangle\right) f d\zeta, \quad z \in D.$$