Generation of Lumps through the Singular Manifold Method

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Summary

- An equation in $2+1$. Motivation
- Singular Manifold Method
- Lax pair
- Darboux transformations
- Lumps
- Expansion in poles of the eigenfuntion

The system: (Fokas. Inv. Prob. 1994)

$$m_y + u\omega = 0$$

$$iu_t + u_{xx} + 2um_x = 0, \qquad -i\omega_t + \omega_{xx} + 2\omega m_x = 0$$

- Real version Chakravarty et al, *J. Math. Phys.* (1995)

  Radha and Lakshmanan, *J. Math. Phys.* (1997). Painlevé property

- Complex version Maccari, *J. Math. Phys.* (1996)

  Porsezian, *J. Math. Phys.* (1997). Painlevé property

Painlevé test and Singular Manifold Method

- The Painlevé test requires that all the solutions of the PDE could be locally written as:

$$u(z_1, ...z_n) = \sum_{j=0}^{\infty} u_j(z_1, ...z_n) \, [\phi(z_1, ...z_n)]^{j-a}$$

  where a is an integer positive number and $\phi(z_1, ...z_n)$ a totally arbitrary function.

- Once the Painlevé test has been checked for a given PDE, the Singular Manifold Method (SMM) allows us to derive Bäcklund transformations, Lax pair, Darboux transformations and tau-functions for the PDE.

The Singular Manifold Method

Summary of the results of P.G. Estévez J. Math. Phys. (1998)

- 1) Truncated expansion

$$\begin{aligned} u^{(1)} &= u^{(0)} + A\frac{\phi_x}{\phi} \\ \omega^{(1)} &= \omega^{(0)} + B\frac{\phi_x}{\phi} \\ m^{(1)} &= m^{(0)} + \frac{\phi_x}{\phi} \end{aligned}$$

$\phi(x, y, t) = 0$ is the **singular manifold**

- 2) Useful quantities

$$v = \frac{\phi_{xx}}{\phi_x}$$
$$q = \frac{\phi_y}{\phi_x}$$
$$r = \frac{\phi_t}{\phi_x}$$

- 3) Singular manifold equations

$$q = AB$$
$$r_x = \frac{1}{2}\Big(\frac{iA_{xx}}{A} - i\frac{B_{xx}}{B} - \frac{A_t}{A} - \frac{B_t}{B}\Big)$$
$$s = -\frac{A_{xx}}{A} - \frac{B_{xx}}{B} - i\frac{A_t}{A} + i\frac{B_t}{B} - \frac{r^2}{2} + \int r_t dx$$

- 4) Seed solutions

$$u^{(0)} = -\frac{A}{2}\left(ir + v + \frac{A_x}{A}\right)$$

$$\omega^{(0)} = -\frac{B}{2}\left(-ir + v + \frac{B_x}{B}\right)$$

$$m_x^{(0)} = -\frac{1}{4}\left(2v_x + \frac{A_{xx}}{A} + \frac{B_{xx}}{B} + i\frac{A_t}{A} - i\frac{B_t}{B}\right)$$

Lax pair

The above equations can be linearized by introducing two functions ψ and φ defined as:

$$
\begin{aligned}
v &= \frac{\psi_x}{\psi} + \frac{\varphi_x}{\varphi} \\
r &= i\left(\frac{\psi_x}{\psi} - \frac{\varphi_x}{\varphi}\right)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
A &= -\frac{1}{\omega^{(0)}}\frac{\varphi_y}{\varphi} \\
B &= -\frac{1}{u^{(0)}}\frac{\psi_y}{\psi} \\
q &= \frac{1}{u^{(0)}\omega^{(0)}}\frac{\psi_y}{\psi}\frac{\varphi_y}{\varphi}
\end{aligned}
$$

Iteration procedure

1) For a given seed solution $\{u^{(0)}, \omega^{(0)}, m^{(0)}\}$, determine its eigenfunctions ψ and χ

$$u^{(0)}\psi_{xy} - u_x^{(0)}\psi_y - \left(u^{(0)}\right)^2 \omega^{(0)}\psi = 0$$

$$\omega^{(0)}\varphi_{xy} - \omega_x^{(0)}\varphi_y - \left(\omega^{(0)}\right)^2 u^{(0)}\varphi = 0$$

$$i\psi_t + \psi_{xx} + 2m_x^{(0)}\psi = 0$$

$$-i\varphi_t + \varphi_{xx} + 2m_x^{(0)}\varphi = 0$$

2) The singular manifold can be determined from ψ and χ as:

$$d\phi = \psi\varphi dx + \frac{1}{u^{(0)}\omega^{(0)}}\psi_y\varphi_y\, dy + i\left(\varphi\psi_x - \psi\varphi_x\right) dt$$

3) A new solution can be constructed

$$
\begin{aligned}
u^{(1)} &= u^{(0)} - \frac{1}{\omega^{(0)}} \frac{\psi\,\varphi_y}{\phi} \\
\omega^{(1)} &= \omega^{(0)} - \frac{1}{u^{(0)}} \frac{\varphi\,\psi_y}{\phi} \\
m^{(1)} &= m^{(0)} + \frac{\phi_x}{\phi}
\end{aligned}
$$

Darboux Transformations

Iteration of the eigenfunction: Let ψ_1, φ_1 and ψ_2, φ_2 be two different couples of eigenfunctions for the seed Lax pair

$$\begin{aligned}
0 &= u^{(0)}\psi_{j,xy} - u_x^{(0)}\psi_{j,y} - \left(u^{(0)}\right)^2 \omega^{(0)}\psi_j \\
0 &= \omega^{(0)}\varphi_{j,xy} - \omega_x^{(0)}\varphi_{j,y} - \left(\omega^{(0)}\right)^2 u^{(0)}\varphi_j \\
0 &= i\psi_{j,t} + \psi_{j,xx} + 2m_x^{(0)}\psi_j \\
0 &= -i\varphi_{j,t} + \varphi_{j,xx} + 2m_x^{(0)}\varphi_j
\end{aligned}$$

$$j = 1, 2$$

$$d\phi_j = \psi_j\varphi_j dx + \frac{1}{u^{(0)}\omega^{(0)}}\psi_{j,y}\varphi_{j,y}\, dy + i\left(\varphi_j\psi_{j,x} - \psi_j\varphi_{j,x}\right) dt$$

Iteration of the eigenfuntions

We can implement the truncated Painlevé expansion with

$$
\begin{aligned}
u^{(1)} &= u^{(0)} - \frac{1}{\omega^{(0)}}\frac{\psi_1\varphi_{1,y}}{\phi_1} \\
\omega^{(1)} &= \omega^{(0)} - \frac{1}{u^{(0)}}\frac{\varphi_1\psi_{1,y}}{\phi_1} \\
m^{(1)} &= m^{(0)} + \frac{\phi_{1,x}}{\phi_1}
\end{aligned}
$$

$$
\begin{aligned}
\psi^{(1)} &= \psi_2 - \psi_1\frac{\Omega_{1,2}}{\phi_1} \\
\varphi^{(1)} &= \varphi_2 - \varphi_1\frac{\Omega_{2,1}}{\phi_1} \\
\phi^{(1)} &= \phi_2 - \frac{\Omega_{1,2}\Omega_{2,1}}{\phi_1}
\end{aligned}
$$

where

$$d\Omega_{i,j} = \psi_j \varphi_i dx + \frac{1}{u(0)\omega(0)} \psi_{j,y} \varphi_{i,y}\, dy + i \left(\varphi_i \psi_{j,x} - \psi_j \varphi_{i,x} \right) dt$$

Iterated Lax pair

$\psi^{(1)}$ **and** $\varphi^{(1)}$ **are eigenfunctions for the iterated solution** $\{u^{(1)}, \omega^{(1)}, m^{(1)}\}$**. It means that they satisfy the Lax pair**

$$
\begin{aligned}
0 &= u^{(1)}\psi_{xy}^{(1)} - u_x^{(1)}\psi_y^{(1)} - \left(u^{(1)}\right)^2 \omega^{(1)}\psi^{(1)} \\
0 &= \omega^{(1)}\varphi_{xy}^{(1)} - \omega_x^{(1)}\varphi_y^{(1)} - \left(\omega^{(1)}\right)^2 u^{(1)}\varphi^{(1)} \\
0 &= i\psi_t^{(1)} + \psi_{xx}^{(1)} + 2m_x^{(1)}\psi^{(1)} \\
0 &= -i\varphi_t^{(1)} + \varphi_{xx}^{(1)} + 2m_x^{(1)}\varphi^{(1)}
\end{aligned}
$$

Second iteration

We can iterate again to obtain

$$u^{(2)} = u^{(1)} - \frac{1}{\omega^{(1)}}\frac{\psi^{(1)}\varphi_y^{(1)}}{\phi^{(1)}}, \qquad \omega^{(2)} = \omega^{(1)} - \frac{1}{u^{(1)}}\frac{\varphi^{(1)}\psi_y^{(1)}}{\phi^{(1)}}$$

that gives us a second iterated solution

$$u^{(2)} = u^{(0)} - \frac{1}{\omega^{(0)}}\left[\varphi_{1,y}(\psi_1\phi_2 - \psi_2\Omega_{2,1}) + \varphi_{2,y}(\psi_2\phi_1 - \psi_1\Omega_{1,2})\right]\frac{1}{\tau}$$

$$\omega^{(2)} = \omega^{(0)} - \frac{1}{u^{(0)}}\left[\psi_{1,y}(\varphi_1\phi_2 - \varphi_2\Omega_{1,2}) + \psi_{2,y}(\varphi_2\phi_1 - \varphi_1\Omega_{2,1})\right]\frac{1}{\tau}$$

where $\tau = \phi^{(1)}\phi_1 = \phi_1\phi_2 - \Omega_{1,2}\Omega_{2,1}$

Lumps

We can now determine the first and second iteration by using the trivial seed solution $u_0 = 1, \quad \omega_0 = 1$, In this case the Lax pair is:

$$
\begin{aligned}
0 &= \psi_{j,xy} - \psi_j, \quad & 0 &= \varphi_{j,xy} - \varphi_j \\
0 &= i\psi_{j,t} + \psi_{j,xx}, \quad & 0 &= -i\varphi_{j,t} + \varphi_{j,xx}
\end{aligned}
$$

The normal modes are:

$$\psi_j = e^{k_j\, Q(k_j)} \left[\alpha_j + \beta_j\, P(k_j)\right], \qquad \varphi_j = e^{-n_j\, Q(n_j)} \left[\gamma_j + \delta_j\, P(n_j)\right]$$

$k_j\, \alpha_j$, $\beta_j\, \delta_j$ and γ_j are complex constants. P and Q are:

$$P(k_j) = x - \frac{y}{k_j^2} + 2ik_j t, \qquad Q(k_j) = x + \frac{y}{k_j^2} + ik_j t$$

If we are looking for rational solutions (lumps), we need a polynomial expression for the singular manifolds. Therefore, the exponentials should be removed by setting $n_j = k_j$. Essentially, there are three different possibilities:

Lumps of type I

$$\alpha_1 = \gamma_1 = \alpha_2 = \gamma_2 = 1,\ \beta_1 = \delta_1 = \beta_2 = \delta_2 = 0,\ k_1 = k,\ k_2 = -k^*$$

Eigenfunctions

$$\psi_1 = e^{k\,Q} \qquad \varphi_1 = e^{-k\,Q}, \qquad \psi_2 = e^{-k^*Q^*} \qquad \varphi_2 = e^{k^*Q^*}$$

$$Q = x + \frac{y}{k^2} + ikt, \qquad P = x - \frac{y}{k^2} + 2ikt$$

Real and imaginary part:

$$k = a_0 + ib_0, \Longrightarrow P = X + iY$$

$$X = x - \frac{(a_0^2 - b_0^2)}{(a_0^2 + b_0^2)^2}\, y - 2b_0 t, \qquad Y = 2\frac{a_0 b_0}{(a_0^2 + b_0^2)^2}\, y + 2a_0 t$$

Singular Manifolds

$$\phi_1 = P, \qquad \phi_2 = P^*$$

$$\Omega_{1,2} = -\frac{1}{k+k^*}\frac{1}{e^{k\,Q}e^{k^*Q^*}} \qquad \Omega_{2,1} = \frac{1}{k+k^*}e^{k\,Q}e^{k^*Q^*}$$

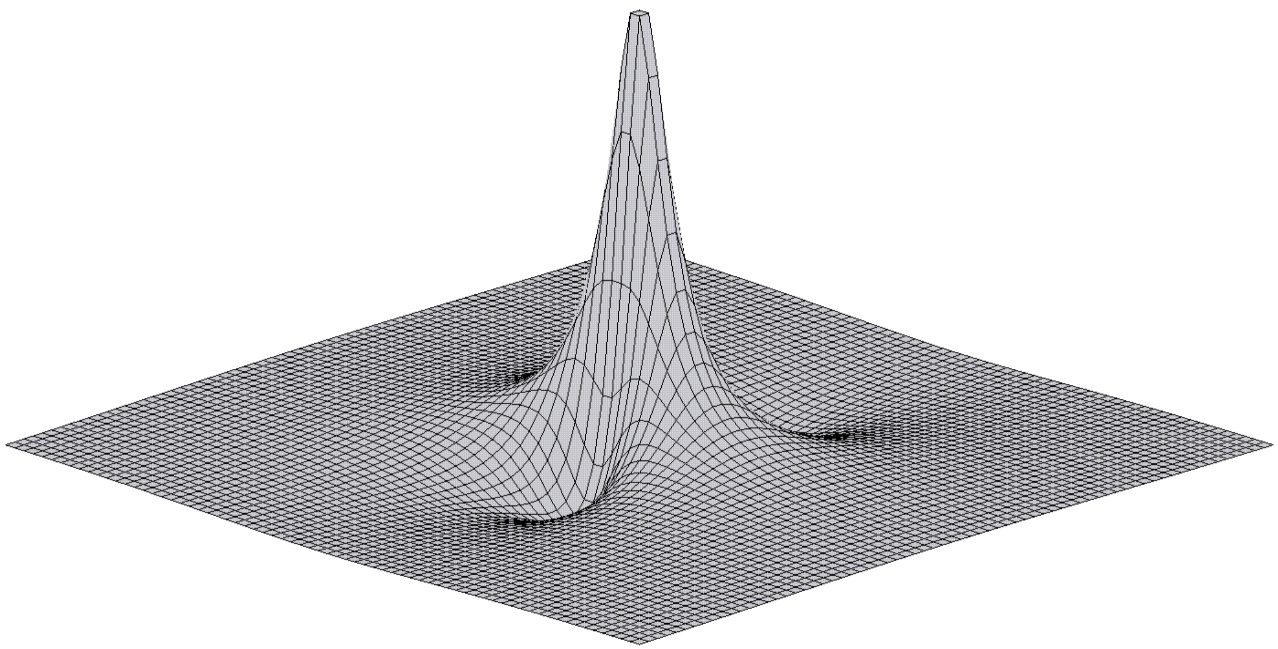
and τ is the positive defined expression

$$\tau = PP^* + \left(\frac{1}{k+k^*}\right)^2 = X^2 + Y^2 + \left(\frac{1}{2a_0}\right)^2$$

that yields the solution

$$u^{(2)} = 1 - \frac{1}{(a_0^2+b_0^2)}\frac{1}{\tau}\left[1 + 2i\left(b_0X + a_0Y\right)\right]$$

$$\omega^{(2)} = 1 - \frac{1}{(a_0^2+b_0^2)}\frac{1}{\tau}\left[1 - 2i\left(b_0X + a_0Y\right)\right]$$



Lumps of type II

$$\alpha_1 = \delta_1 = \beta_2 = \gamma_2 = 1,\ \beta_1 = \gamma_1 = \alpha_2 = \delta_2 = 0,\ k_1 = k,\ k_2 = -k^*$$

Eigenfunctions

$$\psi_1 = e^{k\,Q} \qquad \varphi_1 = e^{-k\,Q}P, \qquad \psi_2 = e^{-k^* Q^*}P^* \qquad \varphi_2 = e^{k^* Q^*}$$

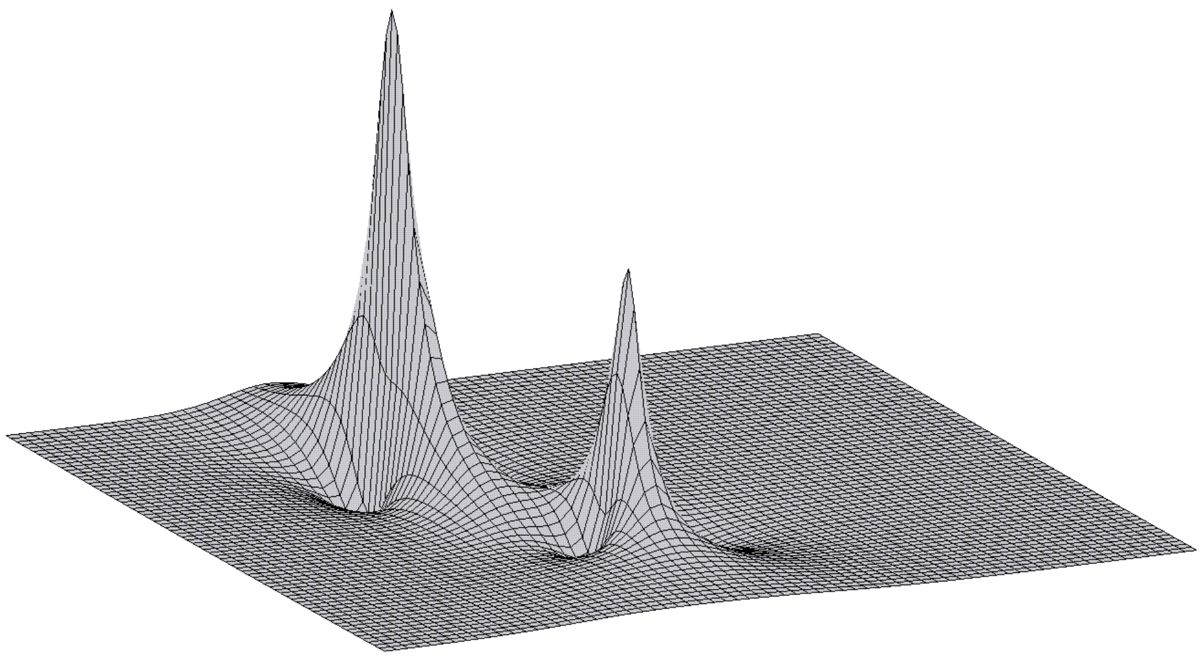
Singular Manifolds

$$\phi_1 = \frac{P^2}{2} - \frac{y}{k^3} - it, \qquad \phi_2 = \phi_1^*$$

$$\begin{aligned}
\Omega_{1,2} &= \left(-\frac{PP^*}{k+k^*} - \frac{P+P^*}{(k+k^*)^2} - \frac{2}{(k+k^*)^3}\right)\frac{1}{e^{k\,Q}e^{k^*\,Q^*}} \\
&= -\frac{1}{2a_0}\left[\left(X+\frac{1}{2a_0}\right)^2 + Y^2 + \frac{1}{4a_0^2}\right]\frac{1}{e^{k\,Q}e^{k^*\,Q^*}} \\
\Omega_{2,1} &= \frac{1}{k+k^*}e^{k\,Q}e^{k^*\,Q^*} = \frac{1}{2a_0}e^{k\,Q}e^{k^*\,Q^*}
\end{aligned}$$

τ is the positive defined expression

$$\tau = \frac{1}{4}\left[X^2 - Y^2 - \frac{1}{b_0}\frac{a_0^2 - 3b_0^2}{a_0^2 + b_0^2}(Y - 2a_0t)\right]^2 + \\ + \left[XY - t + \frac{1}{2a_0}\frac{3a_0^2 - b_0^2}{a_0^2 + b_0^2}(Y - 2a_0t)\right]^2 + \\ + \frac{1}{4a_0^2}\left[\left(X + \frac{1}{2a_0}\right)^2 + Y^2 + \frac{1}{4a_0^2}\right]$$



Lumps of type III

$$\beta_1 = \gamma_1 = \beta_2 = \gamma_2 = 1,\ \alpha_1 = \delta_1 = \alpha_2 = \delta_2 = 0,\ k_1 = k,\ k_2 = -k^*$$

Eigenfunctions

$$\psi_1 = e^{k\,Q} P \quad \varphi_1 = e^{-k\,Q} P, \qquad \psi_2 = e^{-k^* Q^*} P^* \quad \varphi_2 = e^{k^* Q^*} P^*$$

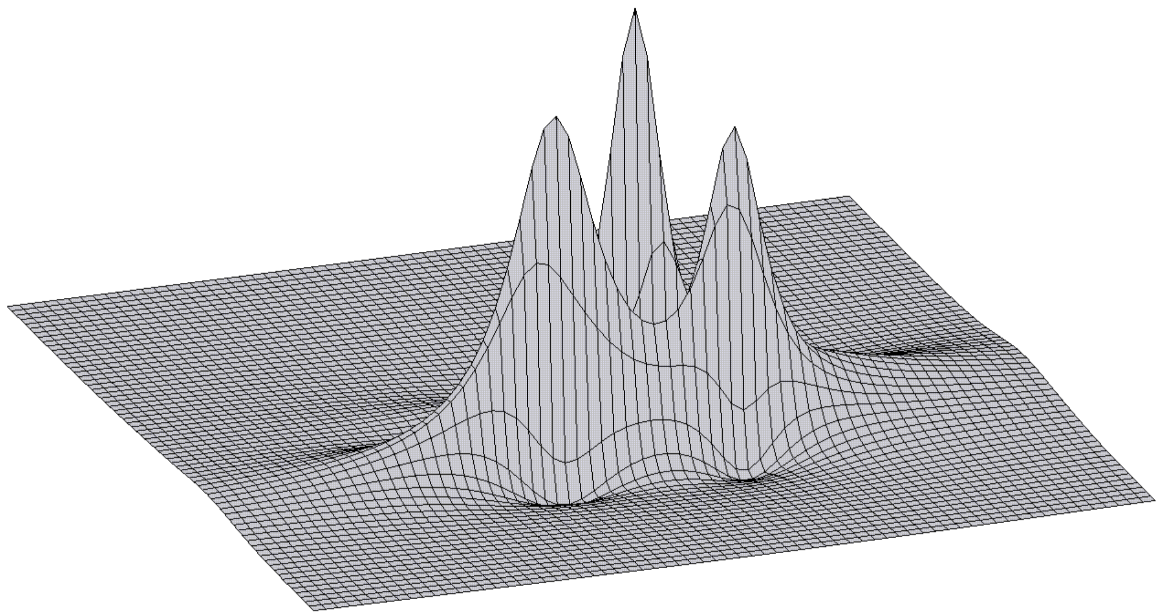
Singular Manifolds

$$
\begin{aligned}
\phi_1 &= \frac{P^3}{3} + \frac{y}{k^4} \\
&= \left[\frac{X^3}{3} - XY^2 + \frac{a_0 b_0}{2(a_0^2 + b_0^2)^2}\left(a_0^4 - 6a_0^2 b_0^2 + b_0^4\right)(Y - 2a_0 t)\right] + \\
&+ i\left[-\frac{Y^3}{3} + X^2 Y - \frac{2}{(a_0^2 + b_0^2)^2}\left(a_0^2 - b_0^2\right)(Y - 2a_0 t)\right] \\
\phi_2 &= \phi_1^*
\end{aligned}
$$

$$
\begin{aligned}
\Omega_{1,2} &= \left(-\frac{PP^*}{k+k^*} - \frac{P+P^*}{(k+k^*)^2} - \frac{2}{(k+k^*)^3}\right)\frac{1}{e^{k\,Q}e^{k^*\,Q^*}} \\
&= -\frac{1}{2a_0}\left[\left(X+\frac{1}{2a_0}\right)^2 + Y^2 + \frac{1}{4a_0^2}\right]\frac{1}{e^{k\,Q}e^{k^*\,Q^*}} \\
\Omega_{2,1} &= \left(\frac{PP^*}{k+k^*} - \frac{P+P^*}{(k+k^*)^2} + \frac{2}{(k+k^*)^3}\right)e^{k\,Q}e^{k^*\,Q^*} \\
&= \frac{1}{2a_0}\left[\left(X-\frac{1}{2a_0}\right)^2 + Y^2 + \frac{1}{4a_0^2}\right]e^{k\,Q}e^{k^*\,Q^*}
\end{aligned}
$$

τ is the positive defined expression

$$\begin{aligned}
\tau \;=\; & \left[\frac{X^3}{3} - XY^2 + \frac{a_0 b_0}{2(a_0^2+b_0^2)^2}\left(a_0^4 - 6a_0^2 b_0^2 + b_0^4\right)(Y - 2a_0 t)\right]^2 + \\
+ \; & \left[-\frac{Y^3}{3} + X^2 Y - \frac{2}{(a_0^2+b_0^2)^2}\left(a_0^2 - b_0^2\right)(Y - 2a_0 t)\right]^2 + \\
+ \; & \frac{1}{4a_0^2}\left[(X^2+Y^2)^2 + \frac{Y^2}{a_0^2} + \frac{1}{4a_0^4}\right]
\end{aligned}$$



Expansion in poles

Ablowitz and Villarroel

A third iteration provides:

$$\psi_3^{(2)} \;=\; \psi_3 - \frac{1}{\tau}\left[\Omega_{1,3}\left(\psi_1\phi_2 - \psi_2\Omega_{2,1}\right) + \Omega_{2,3}\left(\psi_2\phi_1 - \psi_1\Omega_{1,2}\right)\right]$$

that allows us to obtain the second iteration $\psi_3^{(2)}$ of the eigenfunction through the seed eigenfuntions ψ_1, ψ_2, ψ_3

For the Lump case

$$\psi_3^{(2)} = \psi_3 \left(1 + \frac{\nu_3}{(k_3 - k)^3} + \frac{\nu_2}{(k_3 - k)^2} + \frac{\nu_1}{k_3 - k} \right) +$$

$$+\psi_3 \left(\frac{\mu_3}{(k_3 + k^*)^3} + \frac{\mu_2}{(k_3 + k^*)^2} + \frac{\mu_1}{k_3 + k^*} \right)$$

where μ_j and ν_j are easily determined.

Conclusions

- An equation in $2+1$ dimensions is presented and the Painlevé test is successfully applied.
- The SMM is used to construct Darboux transformations and an algorithmic method of deriving solutions
- The above method is used to obtain rationally decaying solutions (LUMPS)
- Connection with other methods of obtaining lumps are presented