On universality of critical behaviour in Hamiltonian PDEs

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Main subject: Hamiltonian perturbations of hyperbolic PDEs

\[ u_t^i + A_j^i(u)u_x^j + \text{higher order derivatives} = 0, \quad i = 1, \ldots, n \]

Weak dispersion expansion: start from

\[ u_t^i + F^i(u, u_x, u_{xx}, \ldots) = 0 \]

Introduce slow variables \( x \mapsto \epsilon x, \quad t \mapsto \epsilon t \)

\[ u_t^i + \frac{1}{\epsilon} F^i(u, \epsilon u_x, \epsilon^2 u_{xx}, \ldots) \]

\[ = u_t^i + A_j^i(u)u_x^j + \epsilon \left( B_j^i(u)u_{xx}^j + \frac{1}{2} C_{jk}^i(u)u_x^j u_x^k \right) + \ldots = 0 \]
The leading term: hyperbolic system (the dispersionless limit)

\[ u_t^i + A_{j}^{i}(u)u_x^j = 0, \quad i = 1, \ldots, n \]

roots of

\[ \det \left( A_{j}^{i}(u) - \lambda \delta_{j}^{i} \right) = 0 \]

are real and pairwise distinct ⇒ local existence theorem.

Globally solutions exist till the time \( t = t_C \)

of gradient catastrophe
The subclass: Hamiltonian perturbations

Main questions:

- Classification
- Properties of solutions
Examples

Example 1 KdV

\[ u_t + u u_x + \frac{\varepsilon^2}{12} u_{xxx} = 0 \]

Example 2 The Volterra lattice (also called difference KdV)

\[ \dot{q}_n = q_n(q_{n+1} - q_{n-1}) \]  \hspace{1cm} (1)

Substitution

\[ q_n = e^{v(n\epsilon)}, \quad t \mapsto 2\epsilon t \]

\[ v_t = \frac{1}{2\epsilon} \left[ e^{v(x+\epsilon)} - e^{v(x-\epsilon)} \right] = e^v v_x + \frac{\epsilon^2}{6} e^v \left( v_x^3 + 3v_x v_{xx} + v_{xxx} \right) + O(\epsilon^4). \]  \hspace{1cm} (2)
**Example 3** Camassa - Holm equation

\[
    u_t = (1 - \epsilon^2 \partial_x^2)^{-1} \left\{ \frac{3}{2} u u_x - \epsilon^2 \left[ u_x u_{xx} + \frac{1}{2} u u_{xxx} \right] \right\} \\
    = \frac{3}{2} u u_x + \epsilon^2 \left( u u_{xxx} + \frac{7}{2} u_x u_{xx} \right) + O(\epsilon^4)
\]

**Example 4** Toda lattice

\[
    \ddot{q}_n = e^{q_{n+1}}q_n - e^{q_n}q_{n-1}.
\]

Continuous version:

\[
    u_n := q_{n+1} - q_n = u(n\epsilon), \quad v_n := \dot{q}_n = v(n\epsilon), \quad t \mapsto \epsilon t
\]

\[
    u_t = \frac{v(x + \epsilon) - v(x)}{\epsilon} = v_x + \frac{1}{2} \epsilon v_{xx} + O(\epsilon^2)
\]

\[
    v_t = \frac{e^{u(x)} - e^{u(x-\epsilon)}}{\epsilon} = e^u u_x - \frac{1}{2} \epsilon (e^u)_{xx} + O(\epsilon^2)
\]
Recall: Hamiltonian hyperbolic systems read

\[ u_t^i + \partial_x \left( \eta_{ij} \frac{\partial h(u)}{\partial u^j} \right) = 0, \quad \eta^{ji} = \eta^{ij}, \quad \det(\eta^{ij}) \neq 0 \]
equivalently

\[ u_t^i + \{u^i(x), H\} = 0, \quad H = \int h(u) \, dx, \quad \{u^i(x), u^j(y)\} = \eta^{ij} \delta'(x-y). \]

**Metric** \[ ds^2 = \eta_{ij} du^i du^j, \quad (\eta_{ij}) = (\eta^{ij})^{-1} \]
Integrable Hamiltonian hyperbolic systems: choice of a system of curvilinear orthogonal coordinates (Tsarev’s theorem)

\[ \tilde{u}^i = \tilde{u}^i(u), \quad ds^2 = \eta_{ij} du^i du^j = \sum_{i=1}^{n} \tilde{\eta}_{ii}(\tilde{u})(d\tilde{u}^i)^2 \]

Parametrized by \( n(n-1)/2 \) arbitrary functions of two variables.

Riemann invariants for the hyperbolic system

\[ \tilde{u}_t^i + a^i(\tilde{u})\tilde{u}_x^i = 0, \quad i = 1, \ldots, n. \]

(Side remark: does a universal dispersionless hierarchy exist?)

Integrability: existence of a complete family of commuting Hamiltonians / complete abelian Lie algebra of symmetries
Deformation problem: given a Hamiltonian hyperbolic system

\[ u_t^i + A_j^i(u)u_x^j = 0 \]

describe all Hamiltonian deformations of the form

\[ u_t^i + A_j^i(u)u_x^j + \epsilon \left( B_j^i(u)u_x^j + \frac{1}{2} C_{jk}^i(u)u_x^j u_x^k \right) + \ldots = 0 \]

In particular:

- what part of symmetries survives after the perturbation?
- How to classify perturbations preserving all symmetries?
- More specific task: classify integrable perturbations of integrable hyperbolic systems
Classify with respect to the group of **Miura-type transformations**

\[ u^i \mapsto \tilde{u}^i = \sum_{k=0}^{\infty} \epsilon^k F_k^i(u, u_x, \ldots, u^{(k)}), \quad i = 1, \ldots, n \]

\[ F_k^i \text{ a polynomial in } u_x, u_{xx}, \ldots, \ deg F_k^i = k, \]

\[ \det \left( \frac{\partial F_0^i(u)}{\partial u^j} \right) \neq 0. \]

**Definition.** The perturbation is called **trivial** if it can be eliminated by a Miura-type transformation.
Lemma  The perturbed Hamiltonian system can be reduced to the form

\[
u_t^i + \{u^i(x), H\} \equiv u_t^i + \partial_x \left( \eta^{ij} \frac{\delta H}{\delta u^j(x)} \right) = 0
\]

\[
H = \int \left[ h_0(u) + \epsilon h_1(u; u_x) + \epsilon^2 h_2(u; u_x, u_{xx}) + \ldots \right] \, dx
\]

\[
\deg h_k(u; u_x, \ldots, u^{(k)}) = k
\]

\Rightarrow \text{ it suffices to classify perturbed Hamiltonians modulo canonical transformations generated by a Hamiltonian } F

\[
u_i \mapsto u^i + \epsilon \{u^i(x), F\} + \frac{\epsilon^2}{2} \{\{u^i(x), F\}, F\} + \ldots
\]

(3)

Used: triviality of the Poisson cohomology of

\[
\{u^i(x), u^j(y)\} = \eta^{ij} \delta'(x - y)
\]

(E.Getzler; F.Magri et al.)
Example 1. Bihamiltonian structure of KdV

\[ u_t = u u_x + \frac{\epsilon^2}{12} u_{xxx} = \{u(x), H_1\}_1 = \frac{3}{2} \{u(x), H_0\}_2 \]

\[ \{u(x), u(y)\}_1 = \delta'(x - y) \]

\[ \{u(x), u(y)\}_2 = u(x)\delta'(x - y) + \frac{1}{2} u_x \delta(x - y) + \frac{\epsilon^2}{8} \delta'''(x - y) \]

\[ H_1 = \int \left( \frac{1}{6} u^3 - \frac{\epsilon^2}{24} u_x^2 \right) dx, \quad H_0 = \int \frac{1}{2} u^2 dx \]
Example 2. Volterra lattice

\[ \dot{q}_n = q_n(q_{n+1} - q_{n-1}). \]

\[ \{q_n, q_m\}_1 = 2q_nq_m(\delta_{n+1,m} - \delta_{n,m+1}) \]

Put \( q_n = e^{v(n\epsilon)} \) \( \Rightarrow \)

\[ H = \frac{1}{2} \int e^v(x) \, dx \]

\[ \{v(x), v(y)\} = \frac{1}{4\epsilon} [\delta(x - y + \epsilon) - \delta(x - y - \epsilon)] = \delta'(x - y) + \frac{\epsilon^2}{3} \delta'''(x - y) + \ldots \] (4)

can be reduced to the canonical form

\[ \{u(x), u(y)\} = \delta'(x - y) \]

by the transformation

\[ u = \sqrt{\frac{\epsilon}{\sinh \epsilon \partial_x}} \quad v = v - \frac{\epsilon^2}{12} v_{xx} + \frac{\epsilon^4}{160} v_{xxxx} + O(\epsilon^6). \]
Main questions:

- What are the geometric properties of the perturbed system? In particular, what part of symmetries of the hyperbolic system survives after the perturbation? How to classify the perturbations that preserve all symmetries of the hyperbolic system?

- What are the properties of solutions of the perturbed system? For what class of initial data one can prove existence results for the perturbed system, at least on the interval $t < t_C$, where $t_C$ is the moment of gradient catastrophe for the hyperbolic system? What are the properties of solutions to the perturbed system near $t = t_C$?

It is clear that the properties of trivial perturbations and their solutions do not differ from the properties of the unperturbed hyperbolic systems.
In the remaining part I will consider the simplest case of Hamiltonian perturbations of the equation

\[ v_t + vv_x = 0 \quad \Leftrightarrow \quad v_t + \{v(x), H_0\} = 0 \] (5)

\[ \{v(x), v(y)\} = \delta'(x - y), \quad H_0 = \int \frac{v^3}{6} \, dx \]

(Cf. Lorenzoni nlin/0108015, Strachan nlin/0205051)

**Lemma 1.** Up to the order \( O(\epsilon^4) \), all Hamiltonian perturbations of (5) can be reduced to the form

\[ u_t + \partial_x \frac{\delta H}{\delta u(x)} = 0 \]

\[ H = \int \left[ \frac{u^3}{6} - \epsilon^2 \frac{c(u)}{24} u_x^2 + \epsilon^4 \left( p(u)u_{xx}^2 + s(u)u_x^4 \right) \right] \, dx \] (6)

where \( c(u), p(u), s(u) \) are arbitrary functions. (The function \( s(u) \) can be eliminated by a Miura-type transform.)
We will now analyze the symmetries of the perturbed system (6). For any \( a(v) \) the Hamiltonian equation

\[
v_s + a(v)v_x = 0 \Leftrightarrow v_s + \{v(x), F_0\} = 0 \tag{7}
\]

\( F_0 = \int f(v) \, dx, \quad f''(v) = a(v) \)

is a (infinitesimal) symmetry of (5),

\[
(v_t)_s = (v_s)_t.
\]

**Exercise** Prove that the family of commuting Hamiltonians is complete, i.e., if \( H = \int h(u; u_x, u_{xx}, \ldots) \, dx \) commutes with all the functionals of the form \( F_0 \) then

\[
h(u; u_x, u_{xx}, \ldots) = g(u) + \partial_x(\ldots).
\]
Lemma 2 For any $f$ the Hamiltonian flow

$$u_s + \partial_x \frac{\delta H_f}{\delta u(x)} = 0, \quad H_f = \int h_f \, dx$$

$$h_f = f - \frac{\epsilon^2}{24} c f''' u_x^2 + \epsilon^4 \left[ \left( p f'''' + \frac{c^2 f^{(4)}}{480} \right) u_{xx}^2 \right. \left. - \left( \frac{cc'' f^{(4)}}{1152} + \frac{cc' f^{(5)}}{1152} + \frac{c^2 f^{(6)}}{3456} + \frac{p' f^{(4)}}{6} + \frac{p f^{(5)}}{6} - s f''' \right) u_x^4 \right]$$

is a symmetry, modulo $O(\epsilon^6)$, of (6). Moreover, the Hamiltonians $H_f$ commute pairwise:

$$\{H_f, H_g\} = O(\epsilon^6)$$

for arbitrary two functions $f(u)$ and $g(u)$. 
Example 1. For $c(u) = \text{const}$, $p(u) = s(u) = 0$ one obtains the KdV equation

$$u_t + u u_x + c \frac{\epsilon^2}{12} u_{xxx} = 0.$$ 

Example 2. For $c(u) = 8u$, $p(u) = \frac{1}{3} u \Rightarrow \text{Camassa-Holm equation.}$

Example 3. The case

$$c(u) = 2, \quad p(u) = -\frac{1}{240}, \quad s(u) = \frac{1}{4320}.$$ 

corresponds to the Volterra lattice.
Bihamiltonian structure: the unperturbed equations

\[ v_t + a(v) v_x = 0 \]

are bihamiltonian w.r.t. the Poisson pencil

\[ \{v(x), v(y)\}_1 = \delta'(x-y), \quad \{v(x), v(y)\}_2 = q(u)\delta'(x-y) + \frac{1}{2}q'(u)u_x\delta(x-y) \quad (9) \]

for an arbitrary function \( q(u) \).

**Lemma** For \( c(u) \neq 0 \) the commuting Hamiltonians admit a unique bihamiltonian structure obtained by a deformation of (9) with

\[ p(u) = \frac{c^2}{960} \left[ 5 \frac{c'}{c} - \frac{q''}{q'} \right], \quad s(u) = 0. \quad (10) \]
Next question: existence of solution for $t < t_C$. We will construct a formal asymptotic solution to (6) (and also to all commuting flows (8)) valid on the entire interval $t < t_C$. The basic idea: find a transformation

$$v \mapsto u = v + O(\epsilon)$$

that transforms all solutions to all unperturbed equations of the form (7) to solutions to the corresponding perturbed equations (8).
**Quasitriviality Theorem** There exists a transformation

\[ v \mapsto u = v + \sum_{k=1}^{4} \epsilon^k F_k(u; u_x, \ldots, u^{(n_k)}), \tag{11} \]

where \( F_k \) are **rational functions** in the derivatives homogeneous of the degree \( k \), that transforms all monotone solutions of (7) to solutions, modulo \( O(\epsilon^6) \), of (8) and vice versa.

Proof. Use the canonical transformation (3) generated by the Hamiltonian

\[ F = \int \left[ \frac{1}{24} \epsilon c(v) v_x (1 - \log v_x) - \epsilon^3 \left( \frac{c^2(v)}{5760} \frac{v_{xx}^3}{v_x^3} - \frac{p(v) v_{xx}^2}{4} \frac{v_x^2}{v_x} \right) \right] dx, \]
that is

\[
v \mapsto u = v + \frac{\epsilon^2}{24} \partial_x \left( c \frac{v_{xx}}{v_x} + c' v_x \right) + \epsilon^4 \partial_x \left[ c^2 \left( \frac{v_{xx}^3}{360 v_x^4} - \frac{7 v_{xx} v_{xxx}}{1920 v_x^3} + \frac{v_{xxxx}}{1152 v_x^2} \right) \right]
\]

\[
+c c' \left( \frac{47 v_{xx}^3}{5760 v_x^3} - \frac{37 v_{xx} v_{xxx}}{2880 v_x^2} + \frac{5 v_{xxxxx}}{1152 v_x} \right)
+c c'' \left( \frac{v_{xxx}}{144} - \frac{v_{xx}^2}{360 v_x} \right)
\]

\[
+ \frac{1}{1152} \left( 7 c' c'' v_x v_{xx} + c''^2 v_x^3 + 6 c c''' v_x v_{xx} + c' c''' v_x^3 + c c^{(4)} v_x^3 \right)
\]

\[
+p \left( \frac{v_{xx}^3}{2 v_x^3} - \frac{v_{xx} v_{xxx}}{v_x^2} + \frac{v_{xxxx}}{2 v_x} \right) + p' v_{xxx} + p'' \frac{v_x v_{xx}}{2}
\]

(12)

In this formula

\[
c = c(v), \quad p = p(v), \quad s(u) = \frac{c(u) c'''(u)}{3456}.
\]
Let us apply the transformation (12) to the unperturbed solutions of (7) obtained by the method of characteristics:

\[ x = a(v) t + b(v) \]  

(13)

for an arbitrary smooth function \( b(v) \). The solution arrives at the point of gradient catastrophe at some \( x = x_0, t = t_0, v = v_0 \). At this point one has

\[ x_0 = a(v_0)t_0 + b(v_0) \]
\[ 0 = a'(v_0)t_0 + b'(v_0) \]
\[ 0 = a''(v_0)t_0 + b''(v_0) \]  

(14)

(inflation point). Let us assume the genericity assumption

\[ \kappa := - \left( a'''(v_0)t_0 + b'''(v_0) \right) \neq 0. \]  

(15)
Applying the quasitriviality transformation to the solution (13) one obtains a divergent series (all terms of the same order). Resummation?

The idea: to apply an appropriate rescaling near the point of gradient catastrophe.

**Example** Let us show that near the point of gradient catastrophe any generic solution to (7) behaves like the graph of cubic root, up to space/time shifts, Galilean transformations and rescalings.
To this end we represent the solution in the form \( x = a(v) t + b(v) \). Introduce the new variables

\[
\begin{align*}
\bar{x} &= x - a_0(t - t_0) - x_0 \\
\bar{t} &= t - t_0 \\
\bar{v} &= v - v_0.
\end{align*}
\]

Here \( a_0 = a(v_0) \). Let us now do the following scaling transformation

\[
\begin{align*}
\bar{x} &\mapsto \lambda \bar{x} \\
\bar{t} &\mapsto \lambda^{\frac{2}{3}} \bar{t} \\
\bar{v} &\mapsto \lambda^{\frac{1}{3}} \bar{v}
\end{align*}
\]  \hspace{1cm} (16)
Substituting in (13) and expanding at $\lambda \to 0$ one obtains, after division by $\lambda$

$$\bar{x} = a'_0 \bar{t} \bar{v} - \frac{1}{6} \kappa \bar{v}^3 + O \left( \lambda^{\frac{1}{3}} \right)$$

where $a'_0 := a'(v_0)$. At short distances $\lambda \to 0$, so we obtain at the limit the universal local formula

$$\bar{x} = a'_0 \bar{t} \bar{v} - \frac{1}{6} \kappa \bar{v}^3.$$
Back to the critical behaviour of solutions to (8): we will use the following

**Theorem** The solutions to

$$u_s + \partial_x \frac{\delta H_f}{\delta u(x)} = 0$$

obtained from

$$x = a(v) t + b(v)$$

by the quasitriviality transformation (12) also satisfy the following fourth order ordinary differential equation depending on the parameter $s$ (“string equation”):

$$x = s \frac{\delta H_f'}{\delta u(x)} + \frac{\delta H_g'}{\delta u(x)} + O(\epsilon^6), \quad f''(u) = a(u), \quad g''(u) = b(u).$$

(17)
Proof. Use the “Galilean symmetry”: the flow

\[ u_s + \partial_x \frac{\delta H_f}{\delta u(x)} \equiv u_s + a(u) u_x + O(\epsilon^2) = 0 \]

commutes with

\[ u_\tau = 1 - s \partial_x \frac{\delta H_{f'}}{\delta u(x)} = 1 - s a'(u) u_x + O(\epsilon^2) \]

Then combine with one of the commuting flows generated by the Hamiltonian \( H_{g'} \).
Remark. All solutions to (8) in the class of formal series in $\epsilon$ can be obtained by the method of Theorem by taking $g = g(u; \epsilon)$.

We are now ready to introduce the special function conjecturally describing the universal critical behaviour of solutions to (8). Let us fix two numbers $\alpha$ and $\kappa \neq 0$. Consider the following fourth order ODE for the function $U = U(X)$ depending on $T$ as on the parameter:

$$X = T U - \left[ \frac{1}{6} U^3 + \frac{1}{24} \left( U'^2 + 2U U'' \right) + \frac{1}{240} U^{IV} \right].$$  \hspace{1cm} (18)
Main Conjecture. 1). The ODE (18) has unique solution $U = U(X; T)$ smooth for all real $X \in \mathbb{R}$ for all values of the parameter $T$.

2). For any solution $v = v(x, t)$ of the form $x = a(v) t + b(v)$ to the unperturbed equation $u_s + a(u) u_x = 0$ with a point of gradient catastrophe at $t = t_0, x = x_0$ there exists a solution to the perturbed PDE (8) defined for $0 \leq t < t_0$ and $x$ sufficiently close to $x_0$ admitting an asymptotic expansion given by the quasitriviality (12).
Let us called the solution **generic** if, along with the condition
\[ \kappa = -(a'''(v_0)t_0 + b'''(v_0)) \neq 0 \]
it also satisfies
\[ c_0 := c(v_0) \neq 0. \]  

(19)

3). The above solution can be extended up to \( t = t_0 \); near the point \((x_0, t_0)\) it behaves in the following way

\[
 u \simeq v_0 + \left( \frac{\epsilon^2 c_0}{\kappa^2} \right)^{1/7} U \left( \frac{x - a_0(t - t_0) - x_0}{(\kappa c_0^3 \epsilon^6)^{1/7}}, \frac{a'_0(t - t_0)}{(\kappa^3 c_0^2 \epsilon^4)^{1/7}} \right) + O(\epsilon^{4/7}).
\]

(20)
“Proof” of the formula (20) is obtained by rescaling

\[ \bar{x} \mapsto \lambda \bar{x} \]

\[ \bar{t} \mapsto \lambda^{\frac{2}{3}} \bar{t} \]  \hspace{1cm} (21)

\[ \bar{v} \mapsto \lambda^{\frac{1}{3}} \bar{v} \]

\[ \epsilon \mapsto \lambda^{7/6} \epsilon. \]

After substitution to the equation (17) and division by \( \lambda \), one obtains
\[ x = a_0' \bar{u} \bar{t} - \kappa \left[ \frac{\bar{u}^3}{6} + \frac{\epsilon^2}{24} c_0 (\bar{u}_x^2 + 2\bar{u} \bar{u}_{xx}) \\
+ \frac{\epsilon^4}{240} c_0^2 \bar{u}_{xxxx} \right] + O\left( \lambda^{1/3} \right). \]

Choosing
\[ \lambda = \epsilon^{6/7} c_0^{3/7} \]
we arrive at the needed asymptotic formula.
**In brief**: all generic solutions of any Hamiltonian perturbations of

\[ v_t + v v_x = 0 \]

have the same, up to shifts, rescalings and Galilean transformations, universal critical behaviour. The same behaviour for the solutions to any of the perturbed commuting flows

\[ v_s + a(v) v_x = 0 \]
The first main difficulty is in proving the first statement of the Conjecture: the existence of the solution to the ODE. This ODE possesses many remarkable properties: the Painlevé property, Lax representation etc. To my best knowledge the conjectural existence of the smooth solution has been first discussed by Brézin, Marinari, Parisi in 1990 (for the particular value $T = 0$ of the parameter) in the setting of the theory of random matrices. The problem remains open so far.
One more example: oscillatory behaviour of correlation functions in the random matrix models:

A more challenging problem is to classify Hamiltonian perturbations of integrable hyperbolic PDEs associated with curvilinear orthogonal coordinates. One of the most interesting question is to describe the perturbations preserving the integrability, i.e., the deformations of maximal abelian subalgebras of Hamiltonian PDEs, and to study their behaviour near the points of gradient catastrophe. These problems are now under careful investigation.