Riemann - Hilbert Problems,
Integrability and Asymptotics
Trieste, September 23, 2005

On universality of critical behaviour in Hamiltonian PDEs

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Main subject: Hamiltonian perturbations of hyperbolic PDEs

$$u^i_t + A^i_j(u)u^j_x + \text{higher order derivatives} = 0, \quad i = 1, \ldots, n$$

Weak dispersion expansion: start from

$$u^i_t + F^i(u, u_x, u_{xx}, \ldots) = 0$$

Introduce **slow variables** $x \mapsto \epsilon\, x, \quad t \mapsto \epsilon\, t$

$$u^i_t + \frac{1}{\epsilon} F^i(u, \epsilon\, u_x, \epsilon^2 u_{xx}, \ldots)$$

$$= u^i_t + A^i_j(u)u^j_x + \epsilon \left(B^i_j(u)u^j_{xx} + \frac{1}{2} C^i_{jk}(u)u^j_x u^k_x \right) + \ldots = 0$$

The leading term: hyperbolic system (the dispersionless limit)

$$u^i_t + A^i_j(u) u^j_x = 0, \quad i = 1, \ldots, n$$

roots of

$$\det\left(A^i_j(u) - \lambda\, \delta^i_j\right) = 0$$

are **real and pairwise distinct** $\Rightarrow$ local existence theorem.

Globally solutions exist till the time $t = t_C$
of **gradient catastrophe**

The subclass: **Hamiltonian perturbations**

Main questions:

- Classification
- Properties of solutions

Examples

Example 1 KdV

$$u_t + u\,u_x + \frac{\varepsilon^2}{12} u_{xxx} = 0$$

Example 2 The Volterra lattice (also called difference KdV)

$$\dot{q}_n = q_n(q_{n+1} - q_{n-1}) \tag{1}$$

Substitution

$$q_n = e^{v(n\epsilon)}, \quad t \mapsto 2\epsilon t$$

$$v_t = \frac{1}{2\epsilon}\left[e^{v(x+\epsilon)} - e^{v(x-\epsilon)}\right] = e^v v_x + \frac{\epsilon^2}{6} e^v (v_x^3 + 3 v_x v_{xx} + v_{xxx}) + O(\epsilon^4). \tag{2}$$

Example 3 Camassa - Holm equation

$$u_t = \left(1 - \varepsilon^2 \partial_x^2\right)^{-1} \left\{ \frac{3}{2} u\, u_x - \varepsilon^2 \left[u_x u_{xx} + \frac{1}{2} u\, u_{xxx} \right] \right\}$$
$$= \frac{3}{2} u\, u_x + \varepsilon^2 \left(u\, u_{xxx} + \frac{7}{2} u_x u_{xx} \right) + O(\varepsilon^4)$$

Example 4 Toda lattice

$$\ddot{q}_n = e^{q_{n+1} - q_n} - e^{q_n - q_{n-1}}.$$

Continuous version:

$$u_n := q_{n+1} - q_n = u(n\varepsilon), \quad v_n := \dot{q}_n = v(n\varepsilon), \quad t \mapsto \varepsilon\, t$$

$$u_t = \frac{v(x+\varepsilon) - v(x)}{\varepsilon} = v_x + \frac{1}{2} \varepsilon\, v_{xx} + O(\varepsilon^2)$$

$$v_t = \frac{e^{u(x)} - e^{u(x-\varepsilon)}}{\varepsilon} = e^u u_x - \frac{1}{2} \varepsilon \left(e^u\right)_{xx} + O(\varepsilon^2)$$

Recall: Hamiltonian hyperbolic systems read

$$u_t^i + \partial_x \left(\eta^{ij} \frac{\partial h(u)}{\partial u^j} \right) = 0, \quad \eta^{ji} = \eta^{ij}, \quad \det(\eta^{ij}) \neq 0$$

equivalently

$$u_t^i + \{u^i(x), H\} = 0, \quad H = \int h(u)\, dx, \quad \{u^i(x), u^j(y)\} = \eta^{ij} \delta'(x-y).$$

Metric $\quad ds^2 = \eta_{ij} du^i du^j, \quad (\eta_{ij}) = (\eta^{ij})^{-1}$

Integrable Hamiltonian hyperbolic systems: choice of a system of **curvilinear orthogonal coordinates** (Tsarev's theorem)

$$\tilde{u}^i = \tilde{u}^i(u), \quad ds^2 = \eta_{ij} du^i du^j = \sum_{i=1}^{n} \tilde{\eta}_{ii}(\tilde{u})(d\tilde{u}^i)^2$$

Parametrized by $n(n-1)/2$ arbitrary functions of two variables.

Riemann invariants for the hyperbolic system

$$\tilde{u}^i_t + a^i(\tilde{u})\tilde{u}^i_x = 0, \quad i = 1, \ldots, n.$$

(**Side remark**: does a **universal** dispersionless hierarchy exist?)
Integrability: existence of a complete family of commuting Hamiltonians / complete abelian Lie algebra of symmetries

Deformation problem: given a Hamiltonian hyperbolic system

$$\mathbf{u_t^i} + \mathbf{A_j^i(u)u_x^j} = 0$$

describe all Hamiltonian deformations of the form

$$\mathbf{u_t^i} + \mathbf{A_j^i(u)u_x^j} + \epsilon \left(B_j^i(u) u_{xx}^j + \frac{1}{2} C_{jk}^i(u) u_x^j u_x^k \right) + \ldots = 0$$

In particular:

- what part of symmetries survives after the perturbation?
- How to classify perturbations preserving all symmetries?
- More specific task: classify *integrable perturbations* of integrable hyperbolic systems

Classify with respect to the group of **Miura-type tranformations**

$$u^i \mapsto \tilde{u}^i = \sum_{k=0}^{\infty} \epsilon^k F_k^i(u; u_x, \ldots, u^{(k)}), \quad i = 1, \ldots, n$$

$$F_k^i \quad \text{a polynomial in} \quad u_x,\, u_{xx}, \ldots, \quad \deg F_k^i = k,$$

$$\det\left(\frac{\partial F_0^i(u)}{\partial u^j}\right) \neq 0.$$

Definition. The perturbation is called **trivial** if it can be eliminated by a Miura-type transformation

Lemma The perturbed Hamiltonian system can be reduced to the form

$$u^i_t + \{u^i(x), H\} \equiv u^i_t + \partial_x \left(\eta^{ij} \frac{\delta H}{\delta u^j(x)} \right) = 0$$

$$H = \int \left[h_0(u) + \epsilon\, h_1(u; u_x) + \epsilon^2 h_2(u; u_x, u_{xx}) + \ldots \right] dx$$

$$\deg h_k(u; u_x, \ldots, u^{(k)}) = k$$

$\Rightarrow$ it suffices to classify perturbed Hamiltonians modulo canonical transformations generated by a Hamiltonian F

$$u^i \mapsto u^i + \epsilon\, \{u^i(x), F\} + \frac{\epsilon^2}{2} \{\{u^i(x), F\}, F\} + \ldots \tag{3}$$

Used: triviality of the Poisson cohomology of

$$\{u^i(x), u^j(y)\} = \eta^{ij} \delta'(x - y)$$

(E.Getzler; F.Magri *et al.*)

Example 1. Bihamiltonian structure of KdV

$$u_t = u\,u_x + \frac{\epsilon^2}{12} u_{xxx} = \{u(x), H_1\}_1 = \frac{3}{2}\{u(x), H_0\}_2$$

$$\{u(x), u(y)\}_1 = \delta'(x-y)$$

$$\{u(x), u(y)\}_2 = u(x)\delta'(x-y) + \frac{1}{2} u_x \delta(x-y) + \frac{\epsilon^2}{8}\delta'''(x-y)$$

$$H_1 = \int \left(\frac{1}{6}u^3 - \frac{\epsilon^2}{24}u_x^2\right) dx, \quad H_0 = \int \frac{1}{2}u^2\, dx$$

Example 2. Volterra lattice

$$\dot{q}_n = q_n(q_{n+1} - q_{n-1}).$$

$$\{q_n, q_m\}_1 = 2q_n q_m(\delta_{n+1,m} - \delta_{n,m+1})$$

Put $q_n = e^{v(n\epsilon)}$ $\Rightarrow$

$$H = \frac{1}{2}\int e^{v(x)}\,dx$$

$$\{v(x), v(y)\} = \frac{1}{4\epsilon}\left[\delta(x-y+\epsilon) - \delta(x-y-\epsilon)\right] = \delta'(x-y) + \frac{\epsilon^2}{3}\delta'''(x-y) + \dots \quad (4)$$

can be reduced to the canonical form

$$\{u(x), u(y)\} = \delta'(x-y)$$

by the transformation

$$u = \sqrt{\frac{\epsilon\partial_x}{\sinh\epsilon\partial_x}}\,v = v - \frac{\epsilon^2}{12}v_{xx} + \frac{\epsilon^4}{160}v_{xxxx} + O(\epsilon^6).$$

Main questions:

• What are the **geometric properties** of the perturbed system? In particular, what part of symmetries of the hyperbolic system survives after the perturbation? How to classify the perturbations that preserve all symmetries of the hyperbolic system?

• What are the properties of **solutions** of the perturbed system? For what class of initial data one can prove existence results for the perturbed system, at least on the interval $t < t_C$, where t_C is the moment of gradient catastrophe for the hyperbolic system? What are the properties of solutions to the perturbed system near $t = t_C$?

It is clear that the properties of **trivial** perturbations and their solutions do not differ from the properties of the unperturbed hyperbolic systems.

In the remaining part I will consider the simplest case of Hamiltonian perturbations of the equation

$$v_t + v\, v_x = 0 \quad \Leftrightarrow \quad v_t + \{v(x), H_0\} = 0 \tag{5}$$
$$\{v(x), v(y)\} = \delta'(x-y), \quad H_0 = \int \frac{v^3}{6}\, dx$$

(Cf. Lorenzoni nlin/0108015, Strachan nlin/0205051)

Lemma 1. Up to the order $O(\epsilon^4)$, all Hamiltonian perturbations of (5) can be reduced to the form

$$u_t + \partial_x \frac{\delta H}{\delta u(x)} = 0$$
$$H = \int \left[\frac{u^3}{6} - \epsilon^2 \frac{c(u)}{24} u_x^2 + \epsilon^4 \left(p(u) u_{xx}^2 + s(u) u_x^4 \right) \right] dx \tag{6}$$

where $c(u)$, $p(u)$, $s(u)$ are arbitrary functions. (The function $s(u)$ can be eliminated by a Miura-type transform.)

We will now analyze the symmetries of the perturbed system (6). For any $a(v)$ the Hamiltonian equation

$$\begin{aligned} &v_s + a(v)v_x = 0 \quad \Leftrightarrow v_s + \{v(x), F_0\} = 0 \\ &F_0 = \int f(v)\,dx, \quad f''(v) = a(v) \end{aligned} \tag{7}$$

is a (infinitesimal) symmetry of (5),

$$(v_t)_s = (v_s)_t.$$

Exercise Prove that the family of commuting Hamiltonians is **complete**, i.e., if $H = \int h(u; u_x, u_{xx}, \ldots)\,dx$ commutes with all the functionals of the form F_0 then

$$h(u; u_x, u_{xx}, \ldots) = g(u) + \partial_x(\ldots).$$

Lemma 2 For any f the Hamiltonian flow

$$u_s + \partial_x \frac{\delta H_f}{\delta u(x)} = 0, \quad H_f = \int h_f \, dx$$

$$h_f = f - \frac{\epsilon^2}{24} c \, f''' u_x^2 + \epsilon^4 \left[\left(p \, f''' + \frac{c^2 \, f^{(4)}}{480} \right) {u_{xx}}^2 \right. \tag{8}$$

$$\left. - \left(\frac{c \, c'' \, f^{(4)}}{1152} + \frac{c \, c' \, f^{(5)}}{1152} + \frac{c^2 \, f^{(6)}}{3456} + \frac{p' \, f^{(4)}}{6} + \frac{p \, f^{(5)}}{6} - s \, f''' \right) {u_x}^4 \right]$$

is a symmetry, modulo $O(\epsilon^6)$, of (6). Moreover, the Hamiltonians H_f commute pairwise:

$$\{H_f, H_g\} = O(\epsilon^6)$$

for arbitrary two functions $f(u)$ and $g(u)$.

Example 1. For $c(u) = \text{const}$, $p(u) = s(u) = 0$ one obtains the KdV equation

$$u_t + u\,u_x + c\frac{\epsilon^2}{12}u_{xxx} = 0.$$

Example 2. For $c(u) = 8\,u$, $p(u) = \frac{1}{3}u \;\Rightarrow$ Camassa-Holm equation.

Example 3. The case

$$c(u) = 2, \quad p(u) = -\frac{1}{240}, \quad s(u) = \frac{1}{4320}.$$

corresponds to the Volterra lattice.

Bihamiltonian structure: the unperturbed equations

$$v_t + a(v)\, v_x = 0$$

are bihamiltonian w.r.t. the Poisson pencil

$$\{v(x), v(y)\}_1 = \delta'(x-y), \quad \{v(x), v(y)\}_2 = q(u)\delta'(x-y) + \frac{1}{2} q'(u) u_x \delta(x-y) \quad (9)$$

for an arbitrary function $q(u)$.

Lemma For $c(u) \neq 0$ the commuting Hamiltonians admit a unique bihamiltonian structure obtained by a deformation of (9) with

$$p(u) = \frac{c^2}{960}\left[5\,\frac{c'}{c} - \frac{q''}{q'}\right], \quad s(u) = 0. \qquad (10)$$

Next question: existence of solution for $t < t_C$. We will construct a **formal asymptotic solution** to (6) (and also to all commuting flows (8)) valid on the entire interval $t < t_C$. The basic idea: find a transformation

$$v \mapsto u = v + O(\epsilon)$$

that transforms **all** solutions to **all unperturbed equations** of the form (7) to solutions to the corresponding **perturbed equations** (8).

Quasitriviality Theorem There exists a transformation

$$v \mapsto u = v + \sum_{k=1}^{4} \epsilon^k F_k(u; u_x, \ldots, u^{(n_k)}), \tag{11}$$

where F_k are **rational functions** in the derivatives homogeneous of the degree k, that transforms all monotone solutions of (7) to solutions, modulo $O(\epsilon^6)$, of (8) and vice versa.

Proof. Use the canonical transformation (3) generated by the Hamiltonian

$$F = \int \left[\frac{1}{24} \epsilon\, c(v)\, v_x (1 - \log v_x) - \epsilon^3 \left(\frac{c^2(v)}{5760} \frac{v_{xx}^3}{v_x^3} - \frac{p(v)}{4} \frac{v_{xx}^2}{v_x} \right) \right] dx,$$

that is

$$
\begin{aligned}
v \mapsto u = v &+ \frac{\epsilon^2}{24}\partial_x\left(c\frac{v_{xx}}{v_x} + c' v_x\right) + \epsilon^4 \partial_x \left[c^2 \left(\frac{v_{xx}^3}{360\, v_x^4} - \frac{7\, v_{xx} v_{xxx}}{1920\, v_x^3} + \frac{v_{xxxx}}{1152\, v_x^2} \right)_x \right. \\
&+ c\, c' \left(\frac{47\, {v_{xx}}^3}{5760\, {v_x}^3} - \frac{37\, v_{xx}\, v_{xxx}}{2880\, {v_x}^2} + \frac{5\, v_{xxxx}}{1152\, v_x} \right) + {c'}^2 \left(\frac{v_{xxx}}{384} - \frac{{v_{xx}}^2}{5760\, v_x} \right) \\
&+ c\, c'' \left(\frac{v_{xxx}}{144} - \frac{{v_{xx}}^2}{360\, v_x} \right) \\
&+ \frac{1}{1152} \left(7\, c'\, c''\, v_x\, v_{xx} + {c''}^2\, {v_x}^3 + 6\, c\, c'''\, v_x\, v_{xx} + c'\, c'''\, {v_x}^3 + c\, c^{(4)}\, {v_x}^3 \right) \\
&\left. + p \left(\frac{{v_{xx}}^3}{2\, {v_x}^3} - \frac{v_{xx}\, v_{xxx}}{{v_x}^2} + \frac{v_{xxxx}}{2\, v_x} \right) + p' v_{xxx} + p'' \frac{v_x\, v_{xx}}{2} \right]
\end{aligned}
\tag{12}
$$

In this formula

$$
c = c(v), \quad p = p(v), \quad s(u) = \frac{c(u)\, c'''(u)}{3456}.
$$

Let us apply the transformation (12) to the unperturbed solutions of (7) obtained by the *method of characteristics*:

$$x = a(v)\,t + b(v) \tag{13}$$

for an arbitrary smooth function $b(v)$. The solution arrives at the point of gradient catastrophe at some $x = x_0$, $t = t_0$, $v = v_0$. At this point one has

$$\begin{aligned} x_0 &= a(v_0)t_0 + b(v_0) \\ 0 &= a'(v_0)t_0 + b'(v_0) \\ 0 &= a''(v_0)t_0 + b''(v_0) \end{aligned} \tag{14}$$

(inflection point). Let us assume the **genericity assumption**

$$\kappa := -\Big(a'''(v_0)t_0 + b'''(v_0)\Big) \neq 0. \tag{15}$$

Applying the quasitriviality transformation to the solution (13) one obtains a divergent series (all terms of the same order). Resummation?

The idea: to apply an appropriate rescaling near the point of gradient catastrophe.

Example Let us show that near the point of gradient catastrophe any generic solution to (7) behaves like the graph of cubic root, up to space/time shifts, Galilean transformations and rescalings.

To this end we represent the solution in the form $x = a(v)\,t + b(v)$. Introduce the new variables

$$\begin{aligned}
\bar{x} &= x - a_0(t - t_0) - x_0 \\
\bar{t} &= t - t_0 \\
\bar{v} &= v - v_0.
\end{aligned}$$

Here $a_0 = a(v_0)$. Let us now do the following scaling transformation

$$\begin{aligned}
\bar{x} &\mapsto \lambda\ \bar{x} \\
\bar{t} &\mapsto \lambda^{\frac{2}{3}}\,\bar{t} \\
\bar{v} &\mapsto \lambda^{\frac{1}{3}}\,\bar{v}
\end{aligned} \tag{16}$$

Substituting in (13) and expanding at $\lambda \to 0$ one obtains, after division by λ

$$\bar{x} = a_0' \bar{t}\, \bar{v} - \frac{1}{6} \kappa\, \bar{v}^3 + O\left(\lambda^{\frac{1}{3}}\right)$$

where $a_0' := a'(v_0)$. At short distances $\lambda \to 0$, so we obtain at the limit the universal local formula

$$\bar{x} = a_0' \bar{t}\, \bar{v} - \frac{1}{6} \kappa\, \bar{v}^3.$$

Back to the critical behaviour of solutions to (8): we will use the following

Theorem The solutions to

$$u_s + \partial_x \frac{\delta H_f}{\delta u(x)} = 0$$

obtained from

$$x = a(v)\, t + b(v)$$

by the quasitriviality transformation (12) also satisfy the following fourth order ordinary differential equation depending on the parameter s ("string equation"):

$$x = s\, \frac{\delta H_{f'}}{\delta u(x)} + \frac{\delta H_{g'}}{\delta u(x)} + O\left(\epsilon^6\right), \quad f''(u) = a(u), \quad g''(u) = b(u). \tag{17}$$

Proof. Use the "Galilean symmetry": the flow

$$u_s + \partial_x \frac{\delta H_f}{\delta u(x)} \equiv u_s + a(u)\, u_x + O(\epsilon^2) = 0$$

commutes with

$$u_\tau = 1 - s\, \partial_x \frac{\delta H_{f'}}{\delta u(x)} = 1 - s\, a'(u) u_x + O(\epsilon^2)$$

Then combine with one of the commuting flows generated by the Hamiltonian $H_{g'}$.

Remark. **All** solutions to (8) in the class of formal series in ϵ can be obtained by the method of Theorem by taking $g = g(u; \epsilon)$.

We are now ready to introduce the special function conjecturally describing the universal critical behaviour of solutions to (8). Let us fix two numbers α and $\kappa \neq 0$. Consider the following fourth order ODE for the function $U = U(X)$ depending on T as on the parameter:

$$X = T\,U - \left[\frac{1}{6}U^3 + \frac{1}{24}\left(U'^2 + 2U\,U''\right) + \frac{1}{240}U^{IV}\right]. \tag{18}$$

Main Conjecture. 1). The ODE (18) has unique solution $U = U(X;T)$ **smooth for all real** $X \in \mathbb{R}$ for all values of the parameter T.

2). For any solution $v = v(x,t)$ of the form $x = a(v)\,t + b(v)$ to the unperturbed equation $u_s + a(u)\,u_x = 0$ with a point of gradient catastrophe at $t = t_0$, $x = x_0$ there exists a solution to the perturbed PDE (8) defined for $0 \leq t < t_0$ and x sufficiently close to x_0 admitting an asymptotic expansion given by the quasitriviality (12).

Let us called the solution **generic** if, along with the condition $\kappa = -\left(a'''(v_0)t_0 + b'''(v_0)\right) \neq 0$ it also satisfies

$$c_0 := c(v_0) \neq 0. \tag{19}$$

3). The above solution can be extended up to $t = t_0$; near the point (x_0, t_0) it behaves in the following way

$$u \simeq v_0 + \left(\frac{\epsilon^2 c_0}{\kappa^2}\right)^{1/7} U\left(\frac{x - a_0(t-t_0) - x_0}{(\kappa\, c_0^3 \epsilon^6)^{1/7}};\ \frac{a_0'(t-t_0)}{(\kappa^3 c_0^2 \epsilon^4)^{1/7}}\right) + O\left(\epsilon^{4/7}\right). \tag{20}$$

"Proof" of the formula (20) is obtained by rescaling

$$\begin{aligned} \bar{x} &\mapsto \lambda\, \bar{x} \\ \bar{t} &\mapsto \lambda^{\frac{2}{3}}\, \bar{t} \\ \bar{v} &\mapsto \lambda^{\frac{1}{3}}\, \bar{v} \\ \epsilon &\mapsto \lambda^{7/6} \epsilon. \end{aligned} \tag{21}$$

After substitution to the equation (17) and division by λ, one obtains

$$\bar{x} = a_0' \bar{u}\, \bar{t} - \kappa \left[\frac{\bar{u}^3}{6} + \frac{\epsilon^2}{24} c_0 \left(\bar{u}_x^2 + 2\bar{u}\, \bar{u}_{xx} \right) \right.$$
$$\left. + \frac{\epsilon^4}{240} c_0^2 \bar{u}_{xxxx} \right] + O\left(\lambda^{1/3}\right).$$

Choosing

$$\lambda = \epsilon^{6/7} c_0^{3/7}$$

we arrive at the needed asymptotic formula.

In brief: **all** generic solutions of **any** Hamiltonian perturbations of

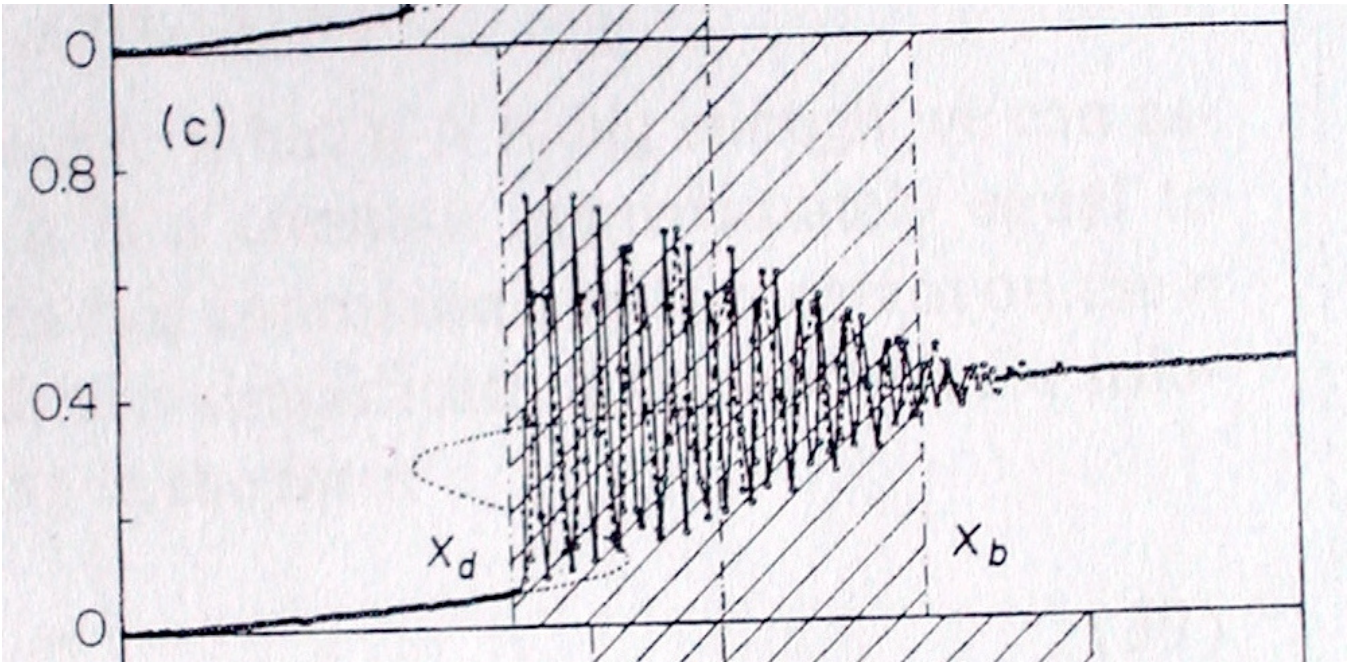
$$v_t + v\, v_x = 0$$

have the **same**, up to shifts, rescalings and Galilean transformations, universal critical behaviour. The same behaviour for the solutions to **any** of the perturbed commuting flows

$$v_s + a(v)\, v_x = 0$$

The first main difficulty is in proving the first statement of the Conjecture: the existence of the solution to the ODE. This ODE posseses many remarkable properties: the Painlevé property, Lax representation etc. To my best knowledge the conjectural existence of the smooth solution has been first discussed by Brézin, Marinari, Parisi in 1990 (for the particular value $T = 0$ of the parameter) in the setting of the theory of random matrices. The problem remains open so far.

One more example: oscillatory behaviour of correlation functions in the random matrix models:



(from Jurkiewicz, Phys. Lett. B, 1991)

A more challenging problem is to classify Hamiltonian perturbations of integrable hyperbolic PDEs associated with curvilinear orthogonal coordinates. One of the most interesting question is to describe the perturbations *preserving the integrability*, i.e., the deformations of maximal abelian subalgebras of Hamiltonian PDEs, and to study their behaviour near the points of gradient catastrophe. These problems are now under careful investigation.