Poisson Geometry
and
Deformation Quantization

joint work with Giovanni Felder

Poisson geometry

Generalization of the Poisson bracket in Hamiltonian mechanics

$$\{f,g\} = \sum_i \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}$$

Def 1 [Poisson algebra]

- $(A, \cdot, \{,\})$ s.t.
  1) $(A, \cdot)$ commutative associative algebra
  2) $(A, \{,\})$ Lie algebra
  3) $\{,\}$ is a biderivation

     i.e. $\{a, bc\} = \{a,b\}c + b\{a,c\}$ $\forall a,b,c$

Def 2 [Poisson manifold]

$(M, \{,\})$: - M smooth manifold
- $\{,\} : C^\infty(M) \otimes C^\infty(M) \to C^\infty(M)$

s.t. $(M, \cdot, \{\})$ Poisson algebra

pointwise product

Remark Since $\{,\}$ is a biderivation,
it is in one-to-correspondence with
a tensor field $\pi \in \Gamma(TM \otimes TM)$

$$\{f,g\} = \pi(df, dg)$$
$$= \pi^{ij}\, \partial_i f\, \partial_j g$$

- skew-symmetry of $\{,\}$ $\Leftrightarrow$ $\pi \in \Gamma(\Lambda^2 TM)$

  $\pi^{ij} = -\pi^{ji}$
  (bivector field)

- Jacobi for $\{,\}$ $\Leftrightarrow$ $[\pi, \pi] = 0$

  Schouten-Nijenhuis bracket

  $$\pi^{i[j} \partial_i \pi^{kl]} = 0$$

Examples

1) $\pi = 0$

2) $M = \mathbb{R}^n$, π^{ij} constant

3) (M, ω) symplectic manifold

$$\{f,g\} = X_f(g), \quad \iota_{X_f}\omega = df$$

4) $M = \mathfrak{g}^*$

$(\mathfrak{g}, [,])$ Lie algebra

$C^\infty(\mathfrak{g}^*) \supset \mathfrak{g}$ linear fns.

Define $\{f, g\} := [f, g] \quad f, g \in \mathfrak{g}$

Extend as a biderivation

or choose a basis of $\mathfrak{g}$, take structure constants f^{ij}_k

define $\pi^{ij}(x) = x^k f^{ij}_k$

5) $M \in \mathbb{R}^2$

$\varphi \in C^\infty(M)$

$\pi = \varphi \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$

Deformation quantization [... Lichnerowicz, Sternheimer]

Generalization of the product induced by Weyl's quantization formula:

$$f \in C^\infty(\underset{q}{\overset{\in}{\mathbb{R}^d}} \times \underset{p}{\overset{\in}{\mathbb{R}^{d*}}}) \rightsquigarrow Q(f) \text{ operator on } L^2(\mathbb{R}^d) \ni \psi$$

$$\left(Q(f)\psi\right)(x) := \int e^{-\frac{i}{\hbar} p\cdot(x-\bar{x})} f\left(p, \frac{x+\bar{x}}{2}\right) \psi(\bar{x}) \frac{d\bar{x}\, dp}{(2\pi\hbar)^d}$$

$$Q(1) = 1$$
$$Q(q) = x\cdot$$
$$Q(p) = -i\hbar \frac{\partial}{\partial x}$$

Define $\quad f * g = \text{"} Q^{-1}\left(Q(f)\cdot Q(g)\right) \text{"}$

either assume $Q(f)\cdot Q(g) \in \operatorname{Im} Q$

or work asymptotically $\hbar \to 0$

$$\Downarrow$$

$$f * g \in C^\infty(\mathbb{R}^d \times \mathbb{R}^{d*})[[\hbar]]$$

formal power series

Explicit formula [Moyal]:

$$f * g = fg + \sum_{n=1}^{\infty} \frac{\varepsilon^n (-1)^n}{n!} \sum_{k+l=n} \frac{(k+l)!}{k!\,l!} \partial_p^l \partial_q^k f \, \partial_p^k \partial_q^l g =$$

$$= fg + \varepsilon \{f, g\} + O(\varepsilon^2) \qquad \varepsilon = \frac{i\hbar}{2}$$

Exponential form

$$f * g(p,q) = \left(e^{\varepsilon\left(\frac{\partial}{\partial q^i}\frac{\partial}{\partial \bar{p}_i} - \frac{\partial}{\partial p_i}\frac{\partial}{\partial \bar{q}^i}\right)} f(p,q)\, g(\bar{p},\bar{q}) \right)\Bigg|_{\substack{\bar{p}=p \\ \bar{q}=q}}$$

Graphically  $= \frac{\partial}{\partial q}\frac{\partial}{\partial \bar{p}} - \frac{\partial}{\partial p}\frac{\partial}{\partial \bar{q}}$

$\sum$ 

<u>In general:</u>

DEFORMATION OF AN COMMUTATIVE ALGEBRA (OF FUNCTIONS)

IN THE CATEGORY OF <u>ASSOCIATIVE ALGEBRAS</u>

Let A be an associative commutative algebra (say, over $\mathbb{R}$)

<u>Def</u> A deformation of A is an associative algebra B over $\mathbb{R}[[\varepsilon]]$ s.t.

1) $B \simeq A[[\varepsilon]]$ as an $\mathbb{R}[[\varepsilon]]$-modules

2) $B/\varepsilon B \xrightarrow[\sim]{\varphi} A$ as $\mathbb{R}$-algebras

[for unital algebras: $1 \in B \mapsto 1 \in A$]

Let $[\,,\,]$ be the commutator in B

Since A is commutative,

$[\,,\,]$ takes values in εB

So we may consider $\frac{1}{\varepsilon}[\,,\,]$

Define

$$\{\underset{A}{a, b}\} := \varphi\left(\frac{1}{\varepsilon}\left[\varphi^{-1}(a), \varphi^{-1}(b)\right]\right)$$

<u>Thm</u> $(A, \cdot, \{,\})$ is a Poisson algebra

<u>Problem</u> Given a Poisson algebra A find a deformation B that gives back the same Poisson bracket

<u>Thm</u> If $A = C^\infty(M)$, M Poisson manifold, this is possible [Kontsevich 97] [Moreover, this is possible in terms of bidifferential operators

[Special case: M symplectic manifold
. De Wilde, Lecomte
. Fedosov
late '80s]

Let B be a deformation of A

A 2-sided ideal $\hat{I}$ of B
induces a Poisson ideal I of A
i.e. I is an ideal in $(A, \cdot)$
and in $(A, \{,\})$

If $A = C^\infty(M)$ and I vanishing ideal of
of a submanifold $S \subset M$,
then S is called a Poisson submanifold.

<u>Problem</u> Is it possible to go the other way around?
That is, given $S \underset{\text{Poisson}}{\subset} M$
is it possible to find a deformation quantization of M
together with an ideal $\hat{I}$ that gives back
the vanishing ideal of S?

Equivalently, $S \overset{i}{\hookrightarrow} M$
is it possible to find def-quant. B, B_S of M, S

Suppose the Poisson bivector field ~~has~~ a zero at $x \in M$.

Then $\{x\}$ is a Poisson submanifold of M.

Solving the problem would mean finding a morphism

$$B \to \mathbb{R}[[\varepsilon]]$$

i.e. associating a character to points where the Poisson algebra is commutative.

This would be quite natural.

But the answer is unknown in general.

One-sided ideals of the deformation B of A induce the so-called coisotropes.

Def A coisotrope I in $(A, \cdot, \{,\})$ is an ideal of $(A, \cdot)$ which is also a Lie subalgebra of $(A, \{,\})$

If $A = C^\infty(M)$, I coisotrope vanishing ideal of $C \subset M$ then C is called a coisotropic submanifold.

Equivalent characterization

Let π be the Poisson bivector field of M

- Let $\pi^\#: T^*M \to TM$

$$(x, \alpha) \mapsto (x, \pi_x(\alpha, \cdot))$$

- Let N^*C conormal bundle of C

$$N^*_x C = \{\alpha \in T^*_x M : \alpha(v) = 0 \ \forall v \in T_x C\}$$
$x \in C$

Then

C is coisotropic $\iff \pi^\#(N^*C) \subset TC$

In the symplectic case this is equivalent to the usual definition $T^\perp C \subset TC$

Let $\hat{I}$ be a one-sided ideal of B (say, a left ideal)

Then $B/\hat{I}$ is a left-module over B

$$\underset{\in B}{b} \cdot \underset{\in B/\hat{I}}{[\beta]} := [b\beta]$$

Let $N(\hat{I}) := \{ b \in B : \varphi b \in \hat{I} \ \forall \varphi \in \hat{I} \}$

Then

1) $N(\hat{I})$ is a subalgebra of B
2) $\hat{I}$ is a 2-sided ideal in $N(\hat{I})$

so $N(\hat{I})/\hat{I}$ is an algebra

3) $B/\hat{I}$ is a right-module over $N(\hat{I})/\hat{I}$

$$B/\hat{I} \ni [\beta] \cdot \underset{\in N(\hat{I})/\hat{I}}{[\tau]} := [\beta \cdot \tau]$$

So $B/\hat{I}$ is a B-module-$N(\hat{I})/\hat{I}$

Let I be the coisotrope induce on A.

Assume $A = C^{\infty}(M)$, I vanishing ideal of C

What is the classical analogue of $N(\hat{I})$?

$$N(I) := \{ f \in A : \{f, \varphi\} \in I \ \forall \varphi \in I \}$$

Lie normalizer

1) $N(I)$ is Poisson subalgebra of A
2) I is a Poisson ideal in N

so $N(I)/I$ is a Poisson algebra

Using the terminology derived from Dirac:

$N(I)$ - first class functions
I - first class constraints

<u>Lemma</u> $N(I)/_I \underset{\substack{\uparrow \\ \text{acts on} \\ \text{commutative} \\ \text{algebra}}}{\cong} \left(A/_I\right)^I = \left(C^{\omega}(C)\right)^I$

. I acts on $A/_I$ by $\{I, \cdot\}$

$C^{\omega}(C)^I = C^{\omega}(\underline{C})$
where $\underline{C}$ is the leaf space of the foliation $\pi^{\#}(N^*C)$

But C may not be smooth.

$$\begin{array}{ccc} C & \hookrightarrow & M \\ \downarrow p & & \\ \underline{C} & & \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} C^\infty(C) & \xleftarrow{i^*} & C^\infty(M) \\ \uparrow p^* & & \\ C^\infty(\underline{C}) & & \end{array}$$

$C^\infty(C)$ is a module over $C^\infty(M)$ and over $C^\infty(\underline{C})$

[Plus Poisson compatibility, induced from: $B/\hat{I}$ is a B-module $N(\hat{I})/\hat{I}$]

Problem: Is it possible to go back?

That is, given a coisotropic submanifold C is it possible to give $C^\infty(C)[[\varepsilon]]$ the structure of a bimodule for the corresponding deformed algebras?

Remark: If $\underline{C}$ is not smooth, it is not even clear that we can find a deformation quantization of $(C^\infty(C))^I$.

Related problem

Let M, N be Poisson manifold

Let $\varphi: M \to N$ be a Poisson map

(i.e. φ^* is a morphism of Poisson algebras)

Problem Is it possible to deform φ^* to a morphism of the deformed algebras?

A solution to this problem would come from quantizing the pair $(\text{graph}\,\varphi, M \times \bar{N})$

where $\text{graph}\,\varphi$ is regarded as a coisotropic submfld of $M \times \bar{N}$ opposite Poisson structure

The Poisson sigma model

Kontsevich's formula for deformation quantization as well as an approach to the problems of the last transparencies comes from the perturbative path integral quantisation of the so-called Poisson sigma model

PSM

- introduced by Ikeda + Schaller - Strobl
- studied (as regards the problems above) by C - Felder

Let (M, π) be a Poisson manifold

disk

Let $\mathcal{M} := \{\text{bundle maps } TD \to T^*M\}$

$$\downarrow$$

$$M^D := \{\text{maps } D \to M\}$$

$$X \in$$

$$\mathcal{M}_X \simeq \Gamma(T^*D \otimes X^*T^*M)$$

The action for $\hat{X} \in \mathcal{M}$

$$\hat{=} (X, \eta) \ , \ \eta \in \mathcal{M}_X$$

is

$$S(\hat{X}) := \int_D \langle \eta, dX \rangle + \frac{1}{2} \langle \eta, \pi^{\#}(X)\eta \rangle$$

where $dX \in \Gamma(T^*D \otimes X^* TM)$

In local coordinates

X^i components of X

$\eta_i \in \Omega^1(D)$ components of η

$$S = \int_D \eta_i \, dX^i + \frac{1}{2} \pi^{ij}(X) \eta_i \eta_j$$

Boundary conditions

- $C \subset M$ submanifold of M

$$X_{|\partial D} : \partial D \to C$$

$$\eta_{|\partial D} \in \Gamma(T^*\partial D \otimes X^* N^* C)$$

$\Rightarrow$ integration by part is well-defined

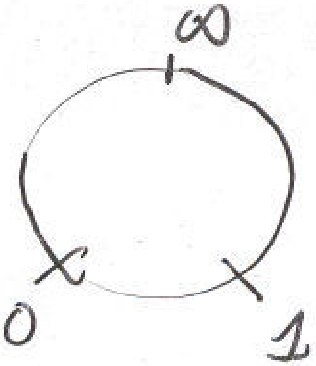
- If C is coisotropic, then one can take $t\pi$ as Poisson structure and the symmetries for the PSM change continuously as t varies (also for $t \to 0$).

$$X \longrightarrow X$$

Special case $C = M$

• Kontsevich formula

$$f*g\,(x) := \int_{\substack{X(\infty)=x \\ c=\eta}} DX\,D\eta \; e^{\frac{i}{\hbar}S}\, f(X(0))\, g(X(1))$$



well-defined (independent of the position of 0 and 1) and associative since the theory is topological.

• Deformation quantization of $\underline{C}$

Use the same formula as above with boundary conditions determined by C.

Expectation: If $f, g \in (C^\infty(C))^I$, $f*g$ should be in $(C^\infty(C))^I[[\varepsilon]]$ and this product should be associative.

Not always true:
Possible anomaly in $H^2_\delta(N^*C)$
Lie algebroid cohomology

If the anomaly vanishes, one can construct the bimodule structure by the same formula but with boundary conditions