String equations and Whitham hierarchies in conformal maps dynamics

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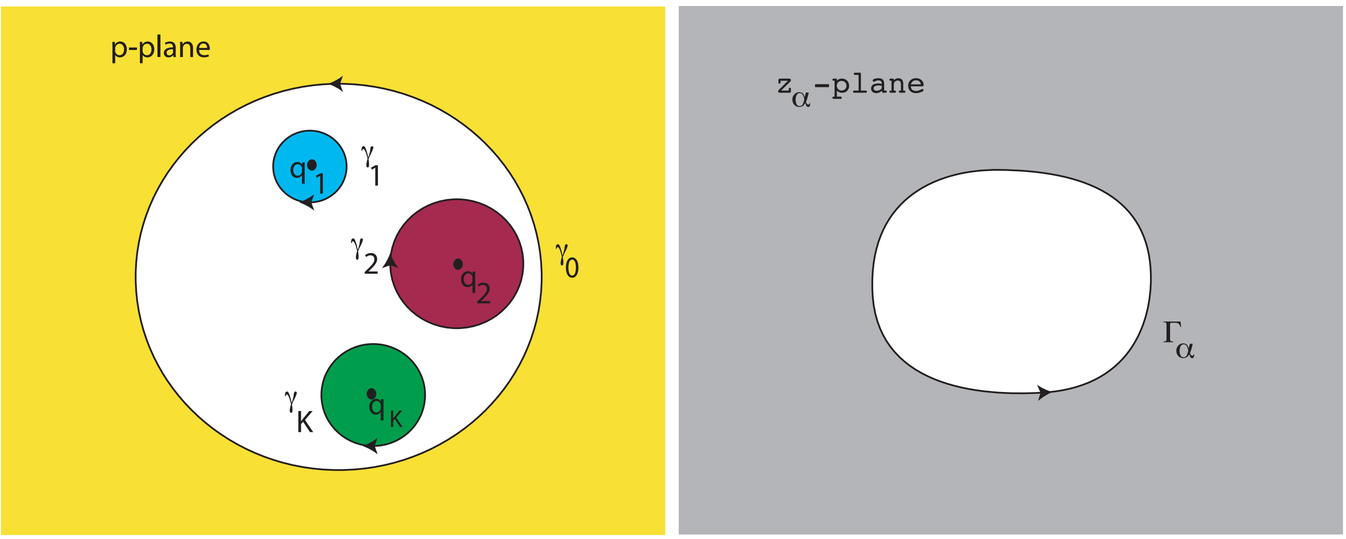
WHITHAM HIERARCHIES (WH_M)

$$\frac{\partial z_\alpha}{\partial t_{\mu n}} = \{\Omega_{\mu n}, z_\alpha\}, \qquad \alpha = 0, \ldots M.$$

$z_\alpha(p)$=Sato functions= local coordinates near $M+1$ *punctures* q_α ($q_0 := \infty$) in the p-plane

$$z_\alpha = \begin{cases} p + \displaystyle\sum_{n=1}^{\infty} \frac{d_{0n}}{p^n}, & \alpha = 0, \\ \displaystyle\frac{d_i}{p - q_i} + \sum_{n=0}^{\infty} d_{in}(p - q_i)^n, & \alpha = i = 1, \ldots, M. \end{cases}$$



- Time parameters

$$\boldsymbol{t} := (t_{\alpha n}), \quad n \geq 0, \quad t_{00} := -\sum_{i=1}^{M} t_{i0}$$

- Poisson bracket

$$\{F, G\} := \frac{\partial F}{\partial p}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial p}, \quad x := t_{01}.$$

- Hamiltonians

$$\Omega_{\alpha n} := \begin{cases} (z_\alpha^n)_{(\alpha,+)}, & n \geq 1, \\ -\log(p - q_i), & n = 0. \end{cases}$$

- m_α = Orlov functions

$$\begin{cases} \{z_\alpha, m_\alpha\} = 1, \quad \forall\alpha, \\ \frac{\partial m_\alpha}{\partial t_{\mu n}} = \{\Omega_{\mu n}, m_\alpha\}, \\ m_\alpha(z_\alpha, \boldsymbol{t}) = \sum_{n=1}^{\infty} n t_{\alpha n} z_\alpha^{n-1} + \frac{t_{\alpha 0}}{z_\alpha} + \sum_{n\geq 2} \frac{v_{\alpha n}}{z_\alpha^n}, \end{cases}$$

String equations

Systems of constraints for (z_α, m_α)

$$\begin{cases} P_i(z_i, m_i) = P_0(z_0, m_0), \\ \\ Q_i(z_i, m_i) = Q_0(z_0, m_0), \end{cases} \qquad i = 1, 2, \ldots, M,$$

where (P_α, Q_α) are pairs of canonical variables (*twistor data*)

$$\{P_\alpha(p, x), Q_\alpha(p, x)\} = 1,$$

THEOREM

Let $(z_\alpha(p, \boldsymbol{t}), m_\alpha(p, \boldsymbol{t}))$ be a solution of a system of string equations such that

$$\mathcal{P}_\alpha(p, \boldsymbol{t}) := P_\alpha(z_\alpha(p, \boldsymbol{t}), m_\alpha(p, \boldsymbol{t})),$$

$$\mathcal{Q}_\alpha(p, \boldsymbol{t}) := Q_\alpha(z_\alpha(p, \boldsymbol{t}), m_\alpha(p, \boldsymbol{t})),$$

are meromorphic functions of the complex variable p with finite poles at $\{q_1, \ldots, q_M\}$ only. Then $(z_\alpha(p, \boldsymbol{t}), m_\alpha(p, \boldsymbol{t}))$ is a solution of the Whitham hierarchy.

Proof. From the hypothesis of the theorem

$$\mathrm{d}z_\alpha \wedge \mathrm{d}m_\alpha = \mathrm{d}\mathcal{P}_\alpha \wedge \mathrm{d}\mathcal{Q}_\alpha, \quad \forall \alpha,$$

$$\mathrm{d}z_\alpha \wedge \mathrm{d}m_\alpha = \mathrm{d}z_\beta \wedge \mathrm{d}m_\beta, \quad \forall \alpha, \beta,$$

$$\mathrm{d}z_\alpha \wedge \mathrm{d}m_\alpha = \sum_{\mu=0}^{M} (\mathrm{d}z_\mu \wedge \mathrm{d}m_\mu)_{(\mu,+)}, \quad \forall \alpha.$$

The terms in these decompositions can be found by using the expansions of the functions m_μ

$$(\mathrm{d}z_\mu \wedge \mathrm{d}m_\mu)_{(\mu,+)}$$

$$= \mathrm{d}\Big(\sum_{n=1}^{\infty} (z_\mu^n)_{(\mu,+)} \mathrm{d}t_{\mu n} - (1 - \delta_{\mu 0}) \log(p - q_\mu) \mathrm{d}t_{\mu 0} \Big)$$

$$= \mathrm{d}\Big(\sum_{n} \Omega_{\mu n} \mathrm{d}t_{\mu n} \Big).$$

Thus we find

$$\mathrm{d}z_\alpha \wedge \mathrm{d}m_\alpha = \mathrm{d}\Big(\sum_{\mu,n} \Omega_{\mu n} \mathrm{d}t_{\mu n} \Big), \quad \forall \alpha,$$

$\square$

Solvable string equations

Given a splitting $\{1,\ldots,M\} = I \cup J$, $I \cap J = \emptyset$, the system of string equations

$$
\boxed{
\begin{gathered}
i \in I \begin{cases} z_i^{n_i} = z_0^{n_0} \\[2ex] \dfrac{1}{n_i}\dfrac{m_i}{z_i^{n_i-1}} = \dfrac{1}{n_0}\dfrac{m_0}{z_0^{n_0-1}} \end{cases} , \\[4ex]
j \in J \begin{cases} -\dfrac{n_0}{n_j}\dfrac{m_j}{z_j^{n_j-1}} = z_0^{n_0} \\[2ex] z_j^{n_j} = \dfrac{m_0}{z_0^{n_0-1}} \end{cases} .
\end{gathered}
}
$$

admits solutions verifying the conditions of the above theorem which can be determined by means of a system of implicit equations.

Scheme of solution

We introduce the ansatz

$$
\begin{aligned}
z_0^{n_0} = z_i^{n_i} = E_1(p) &:= p^{n_0} + \sum_{n=0}^{n_0-2} u_n p^n \\
&+ \sum_{l\in I}\sum_{n=1}^{n_l} \frac{a_{ln}}{(p-q_l)^n} + \sum_{k\in J}\sum_{n=1}^{N_k-n_k} \frac{b_{kn}}{(p-q_k)^n}, \quad \forall i \in I,
\end{aligned}
$$

$$
\begin{aligned}
z_j^{n_j} = E_2(p) &:= \sum_{n=0}^{N_0-n_0} c_n p^n \\
&+ \sum_{l\in I}\sum_{n=1}^{N_l-n_l} \frac{\tilde{a}_{ln}}{(p-q_l)^n} + \sum_{k\in J}\sum_{n=1}^{n_k} \frac{\tilde{b}_{kn}}{(p-q_k)^n}, \quad \forall j \in J,
\end{aligned}
$$

and look for functions m_α of the form

$$
m_\alpha(z, \boldsymbol{t}) = \sum_{n=1}^{N_\alpha} n t_{\alpha n} z_\alpha^{n-1} + \frac{t_{\alpha 0}}{z_\alpha} + \sum_{n\geq 2} \frac{v_{\alpha n}}{z_\alpha^n},
$$

The system of string equations and the required asymptotic conditions lead to

$$
\begin{aligned}
& m_i z_i = \frac{n_i}{n} m_0 z_0, \quad m_j z_j = -\frac{n_j}{n} m_0 z_0, \quad \forall i \in I,\ j \in J, \\
& m_0 z_0 = \sum_{n=1}^{N_0} n t_{0n} (z_0^n)_{(0,+)} + t_{00} + \sum_{i\in I} \frac{n_0}{n_i} \sum_{n=1}^{N_i} n t_{in} (z_i^n)_{(i,+)} \\
& - \sum_{j\in J} \frac{n_0}{n_j} \sum_{n=1}^{N_j} n t_{jn} (z_j^n)_{(j,+)},
\end{aligned}
$$

The problem reduces to solving the system

$$\boxed{\begin{cases} m_0 z_0 = E_1(p)\, E_2(p), \\ \\ Res(m_i, z_i = \infty) = t_{i0}, \quad i = 1, \ldots M - 1. \end{cases}}$$

for the unknowns

$$(q_i, u_n, c_n, a_{ln}, b_{kn}, \tilde{a}_{ln}, \tilde{b}_{kn})$$

in terms of the Whitham times $\boldsymbol{t}$.

Properties

The τ-function is given by

$$2\log\tau = \sum_{\alpha}\sum_{n\geq 0} t_{\alpha n} v_{\alpha n+1} - \sum_{j\in J}\frac{t_{j0}^2}{2\,n_j} - \sum_{j\in J}\frac{1}{n_j}\sum_{n\geq 1} n\, t_{jn} v_{jn+1}.$$

The solutions are invariant under the symmetries generated by

$$\mathbb{V}_{rs} = (P_0^{r+1}\,Q_0^{s+1}, \ldots, P_M^{r+1}\,Q_M^{s+1}), \quad r\geq -1,\ s\geq 0,$$

and satisfy the identities

$$\sum_{\alpha\in\{0\}\cup I}\oint_{\Gamma_\alpha}\left(\frac{z_\alpha}{n_\alpha}\right)^s z_\alpha^{(r-s)n_\alpha} m_\alpha^{s+1}\,\mathrm{d}z_\alpha$$

$$+(-1)^r\frac{s+1}{r+1} n_0^{r-s}\sum_{j\in J}\oint_{\Gamma_j}\left(\frac{z_j}{n_j}\right)^r z_j^{(s-r)n_j} m_j^{r+1}\,\mathrm{d}z_j = 0.$$

Example

$$M=1,\ I=\emptyset,\ n_0=2,\ n_1=1,\ N_0=N_1=3$$

$$z_0^2=p^2+u_0+\frac{a_1}{p-q}+\frac{a_2}{(p-q)^2}, \qquad z_1=\frac{b_1}{p-q}+c_0+c_1 p.$$

One obtains the system

$$
\begin{aligned}
&p^3: && 3t_{03}=c_1,\\
&p^2: && 2t_{02}=c_0,\\
&p^1: && t_{01}+\frac{9t_{03}u_0}{2}=b_1+c_1u_0,\\
&p^0: && \frac{9a_1t_{03}}{2}-t_{10}+2t_{02}u_0=a_1c_1+b_1q+c_0u_0,\\
&(p-q)^{-3}: && -6{b_1}^2t_{13}=a_2,\\
&(p-q)^{-2}: && -2{b_1}^2\left(2t_{12}+9\left(c_0+c_1q\right)t_{13}\right)=\\
& && a_1b_1+a_2\left(c_0+c_1q\right),\\
&(p-q)^{-1}: && -2b_1\Big(t_{11}+4\left(c_0+c_1q\right)t_{12}\\
& && +9\left(c_0^2+2c_0c_1q+c_1b_1+c_1^2q^2\right)t_{13}\Big)=\\
& && a_2c_1+a_1\left(c_0+c_1q\right)+b_1\left(q^2+u_0\right),
\end{aligned}
$$

and by solving these equations we find

$$z_0^2 = p^2$$

$$-\frac{2\left(qt_{01}+t_{10}+6t_{01}t_{03}t_{12}+36t_{01}t_{02}t_{03}t_{13}+54qt_{01}{t_{03}}^2t_{13}\right)}{3t_{03}\left(q+6t_{03}t_{12}+36t_{02}t_{03}t_{13}+54q{t_{03}}^2t_{13}\right)}$$

$$+\frac{4t_{10}\left(t_{12}+6t_{02}t_{13}+9qt_{03}t_{13}\right)}{(p-q)\left(q+54q{t_{03}}^2t_{13}+6t_{03}\left(t_{12}+6t_{02}t_{13}\right)\right)}$$

$$-\frac{6{t_{10}}^2t_{13}}{(p-q)^2\left(q+54q{t_{03}}^2t_{13}+6t_{03}\left(t_{12}+6t_{02}t_{13}\right)\right)^2},$$

$$z_1 = -\frac{t_{10}}{(p-q)\left(q+6t_{03}t_{12}+36t_{02}t_{03}t_{13}+54q{t_{03}}^2t_{13}\right)}$$

$$+2t_{02}+3pt_{03},$$

where q is determined by the implicit equation

$$A(\boldsymbol{t})q^3 + B(\boldsymbol{t})q^2 + C(\boldsymbol{t})q + D(\boldsymbol{t}) = 0,$$

$A(\boldsymbol{t}) := 3t_{03} + 324{t_{03}}^3 t_{13} + 8748{t_{03}}^5 {t_{13}}^2,$

$B(\boldsymbol{t}) := 54{t_{03}}^2 t_{12} + 324 t_{02}{t_{03}}^2 t_{13} + 2916{t_{03}}^4 t_{12}t_{13}$

$+17496 t_{02}{t_{03}}^4 {t_{13}}^2,$

$C(\boldsymbol{t}) := -2t_{01} + 6t_{03}t_{11} + 24t_{02}t_{03}t_{12} + 216{t_{03}}^3{t_{12}}^2$

$+72{t_{02}}^2 t_{03}t_{13} - 108t_{01}{t_{03}}^2 t_{13} + 324{t_{03}}^3 t_{11}t_{13} + 3888t_{02}{t_{03}}^3 t_{12}t_{13}$

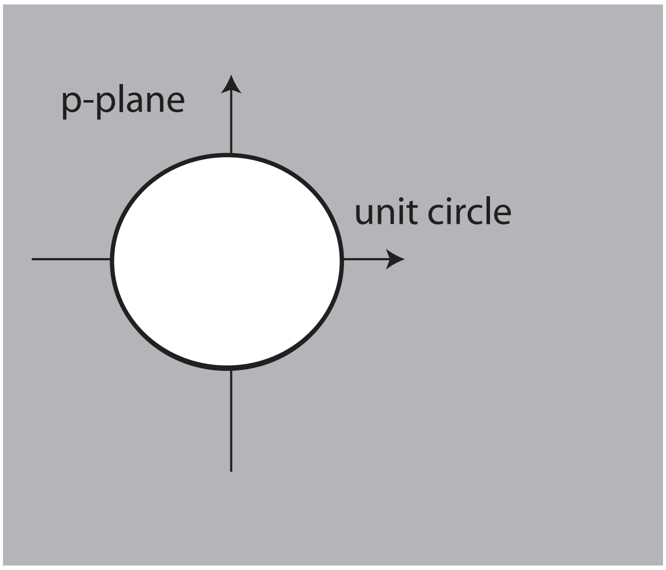
$+11664{t_{02}}^2{t_{03}}^3{t_{13}}^2,$

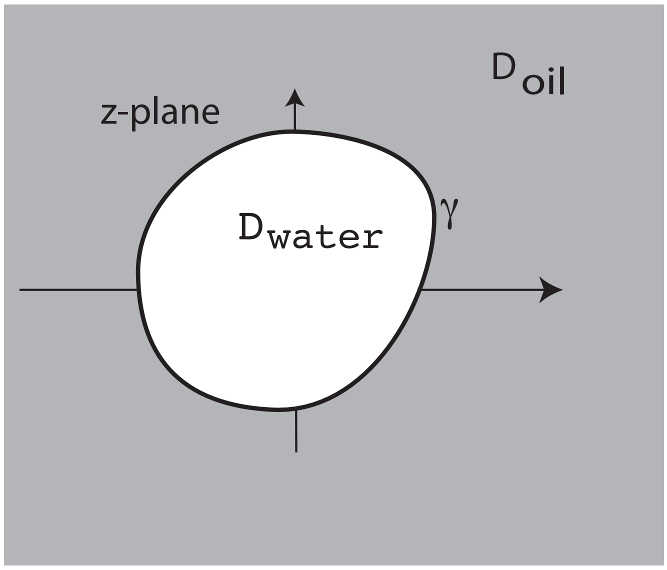
$D(\boldsymbol{t}) := -2t_{10} - 12t_{01}t_{03}t_{12} + +36{t_{03}}^2 t_{11}t_{12}$

$+144t_{02}{t_{03}}^2{t_{12}}^2 - 72t_{01}t_{02}t_{03}t_{13} - 108{t_{03}}^2 t_{10}t_{13} + 216t_{02}{t_{03}}^2 t_{11}t_{13}$

$+1296{t_{02}}^2{t_{03}}^2 t_{12}t_{13} + 2592{t_{02}}^3{t_{03}}^2{t_{13}}^2.$

CONFORMAL MAPS DYNAMICS





D_{water} =simply-connected domain with boundary γ in the z-plane

$z(p)$= conformal map : $\{|p| > 1\} \to D_{oil}$,

$$z(p) = r\,p + \sum_{n=0}^{\infty} \frac{d_n}{p^n}, \quad p \to \infty, \quad r > 0,$$

Observation

Certain types of deformations of $z(p)$ (or γ) are governed by integrable systems.

Example (Wiegmann-Zabrodin (2000))

Suppose that γ is determined by an equation of the form

$$\bar{z} = S(z),$$

where $S(z)$ (Schwarz function of γ) is analytic in a neighborhood of γ.

$t := (t_0, t_1, \ldots) =$ exterior harmonic moments of γ

$$S(z) = \sum_{n\geq 1} n\, t_n\, z^{n-1} + \frac{t_0}{z} + \sum_{n\geq 1} \frac{v_n(t)}{z^n}.$$

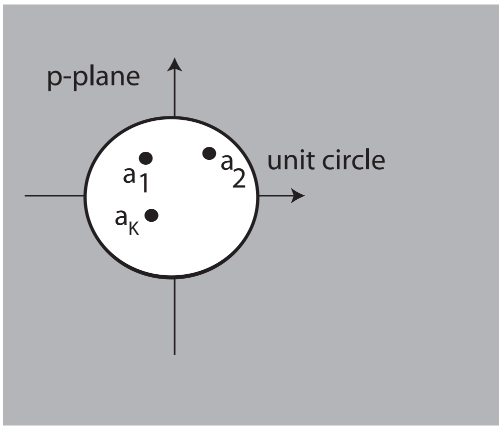
> If the harmonic moments are considered as independent parameters, then $z = z(p,t)$ verifies the dispersionless Toda (dToda) hierarchy.

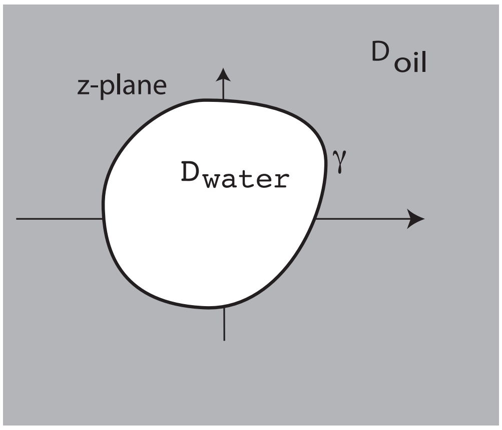
Algebraic domains

Conformal maps $z = z(p)$ given by rational functions

$$z(p) = r\,p + q + \sum_{n=1}^{N_0} \frac{u_n}{p^n} + \sum_{i=1}^{K} \sum_{n=1}^{N_i} \frac{u_{i,n}}{(p - a_i)^n},$$

with K poles $a_i \neq 0$ or orders N_i inside the unit circle.





Problem

$\boxed{\text{To characterize integrable deformations of algebraic domains}}$

The Schwarz function $S = S(z)$ is given by

$$S(z) = \bar{z}(p(z)), \qquad z \in D_{oil}.$$

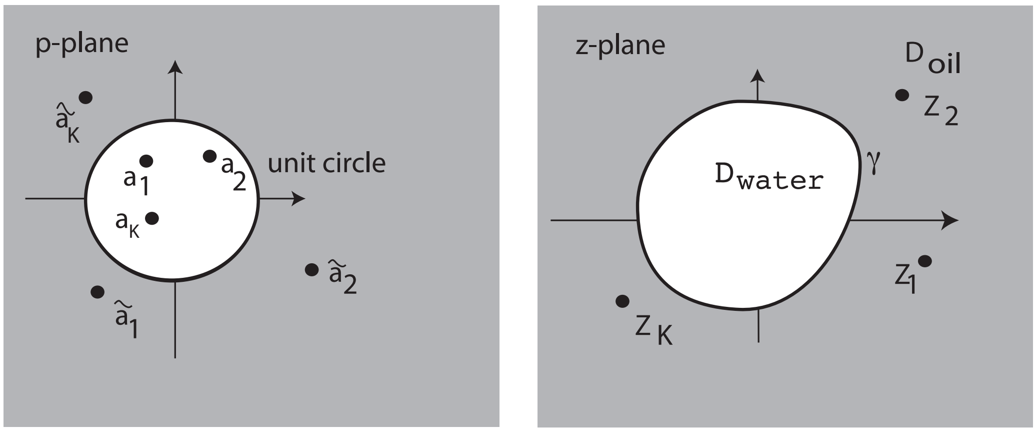
where

$$\bar{z}(p) := \overline{z(\bar{p}^{-1})}$$

$$= \frac{r}{p} + \bar{q} + \sum_{n=1}^{N_0} \bar{u}_n\, p^n + \sum_{i=1}^{K} \sum_{n=1}^{N_i} \frac{\bar{u}_{i,n}\, p^n}{(1 - p\bar{a}_i)^n}.$$

$S(z)$ is meromorphic in D_{oil} with poles at

$$Z_i := z(\tilde{a}_i), \qquad \tilde{a}_i := 1/\bar{a}_i.$$



If $K \geq 1$ then there is an infinite number of nonzero harmonic moments t_n of γ.

Applications to random matrix models

Partition function of normal matrix models in terms of eigenvalues:

$$\begin{cases} Z_N = \int \prod_{i>j} |z_i - z_j|^2 \, e^{\frac{1}{\hbar}\sum_j W(z_j,\bar{z}_j)} \prod_j \mathrm{d}z_j, \\ \\ W(z,\bar{z}) := -z\bar{z} + V(z) + \overline{V(z)} \quad \text{(Potential)} \end{cases}$$

In the large N limit ($N \to \infty$, $\hbar N$ fixed) the eigenvalues densely occupy a domain D in the complex plane (support of eigenvales)

Saddle point approximation:

$$\begin{cases} \bar{z} = V_z(z) + \dfrac{1}{2\pi i} \displaystyle\iint_D \dfrac{\mathrm{d}z' \wedge \mathrm{d}\bar{z}'}{z' - z}, \\ \\ z \in D = D_{water}. \end{cases}$$

For algebraic domains

$V_z(z) = \sum$ Principal parts of $S(z)$ at $\infty, Z_1, \ldots, Z_K$

DYNAMICS OF ALGEBRAIC DOMAINS

From

$$z(p) = r\,p + q + \sum_{n=1}^{N_0} \frac{u_n}{p^n} + \sum_{i=1}^{K} \sum_{n=1}^{N_i} \frac{u_{i,n}}{(p-a_i)^n},$$

we generate solutions of a Whitham hierarchy

$$WH_{2K+1}.$$

.

By introducing the change of variable

$$p_{new} = T\,p_{old} := r\,p_{old} + q,$$

we have

$$z(p) = p + \mathcal{O}(1/p), \qquad p \to \infty.$$

and $z(p)$ and $\bar{z}(p)$ become rational functions with poles at

$$q_1 := q, \qquad q_{i+1} := T\,a_i,$$

and

$$q_1 := q, \qquad \tilde{q}_{i+1} := q + \frac{r^2}{\overline{q_{i+1} - q}},$$

respectively $(i = 1, \ldots, K)$.

Whitham variables

$$\begin{cases} z_0 = z, \quad m_0 = \bar{z}, \quad (q_0 = \infty) \\ z_1 = \bar{z}, \quad m_1 = -z, \quad (q_1) \\ z_{i+1} = z^{1/N_i}, \quad m_{i+1} = N_i \, z_{i+1}^{N_i - 1} \, \bar{z}, \quad (q_{i+1}) \\ \tilde{z}_{i+1} = \bar{z}^{1/N_i}, \quad \tilde{m}_{i+1} = -N_i \, \tilde{z}_{i+1}^{N_i - 1} \, z, \quad (\tilde{q}_{i+1}) \end{cases}$$

String equations

$$I \text{ type} \begin{cases} z_{i+1}^{N_i} = z_0, \\ \dfrac{1}{N_i} \dfrac{m_{i+1}}{z_{i+1}^{N_i-1}} = m_0 \end{cases}$$

$$J \text{ type} \begin{cases} z_1 = m_0, \quad m_1 = -z_0 \\ \begin{cases} \tilde{z}_{i+1}^{N_i} = m_0, \\ -\dfrac{1}{N_i} \dfrac{\tilde{m}_{i+1}}{\tilde{z}_{i+1}^{N_i-1}} = z_0 \end{cases} \end{cases} .$$

Whitham times are introduced by imposing the asymptotic conditions

Time parameters

$$
\begin{cases}
m_0 = \displaystyle\sum_{n=1}^{N_0+1} n t_{0,n} z_0^{n-1} + \frac{t_{0,0}}{z_0} + \cdots \\[2ex]
m_1 = \displaystyle\sum_{n=1}^{N_0+1} n t_{1,n} z_1^{n-1} + \frac{t_{1,0}}{z_1} + \cdots \\[2ex]
m_{i+1} = \displaystyle\sum_{n=1}^{N_i} n t_{i+1,n} z_{i+1}^{n-1} + \frac{t_{i+1,0}}{z_{i+1}} + \cdots \\[2ex]
\tilde{m}_{i+1} = \displaystyle\sum_{n=1}^{N_i} n \tilde{t}_{i+1,n} \tilde{z}_{i+1}^{n-1} + \frac{\tilde{t}_{i+1,0}}{\tilde{z}_{i+1}} + \cdots
\end{cases}
$$

and

$$
t_{1,n} = -\bar{t}_{0,n}, \quad \tilde{t}_{i+1,n} = -\bar{t}_{i+1,n}.
$$

$\boxed{\text{EXAMPLES}}$

1) $K = 0$ (WH_1)

No poles inside the unit circle (except $p = 0$)

$$z(p) = r\,p + q + \sum_{n=1}^{N_0} \frac{u_n}{p^n}.$$

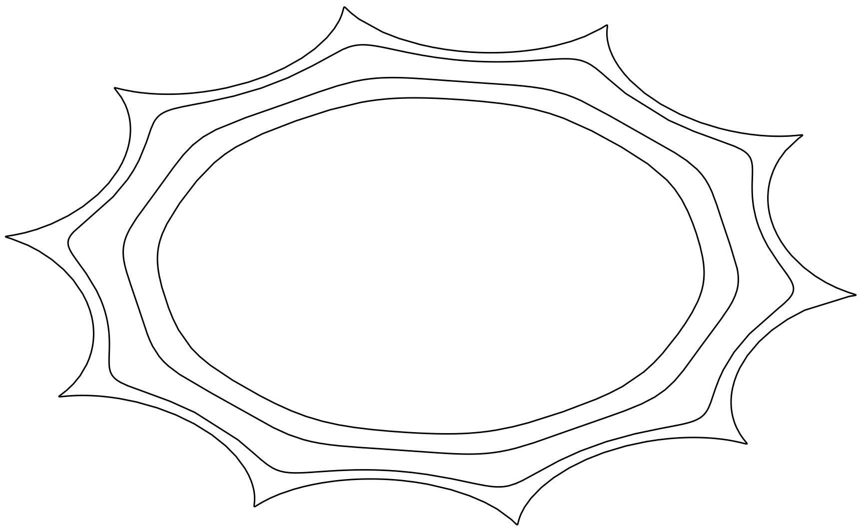
- $S = S(z)$ is holomorphic in D_{oil}.
- $z(p, \boldsymbol{t})$ satisfies the dToda hierarchy.
- The τ-function reduces to the Wiegmann-Zabrodin expression

$$2 \log \tau = \frac{1}{2} \sum_{n \geq 1} (2 - n)(t_n\, v_n + \bar{t}_n\, \bar{v}_n) + t_0\, v_0 - \frac{t_0^2}{2},$$

multicusp solutions

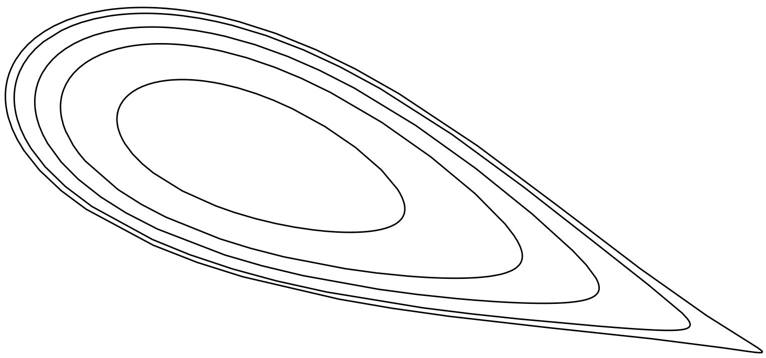
$$\begin{cases} z = r\,p + \dfrac{N\,\bar{t}_N\,r^{N-1}}{p^{N-1}}, \\ \\ N^2(N-1)t_N\bar{t}_N r^{2(N-1)} - r^2 + t_0 = 0. \end{cases}$$

N=10:



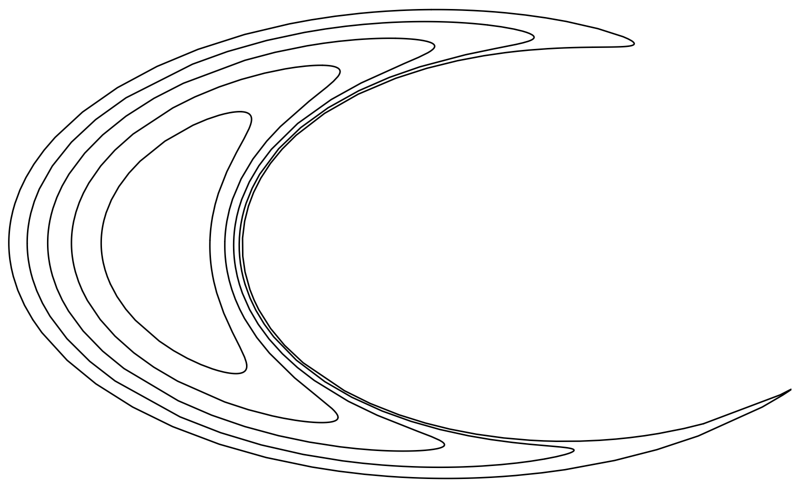
2) $K = 1$ **Aircraft wing** (WH_3)

$$z(p) = r\,p + q + \frac{u}{p - a_1},$$



3) $K = 2$ **Double wing** (WH_5)

$$z(p) = r\,p + q + \frac{u_1}{p - a_1} + \frac{u_2}{p - a_2}.$$



$$t_{01} = q - \frac{u_1}{a_1} - \frac{u_2}{a_2},$$

$$t_{10} = r(\frac{\bar{u}_1}{\bar{a}_1^2} + \frac{\bar{u}_2}{\bar{a}_2^2} - r),$$

$$t_{21} = \bar{q} + \frac{r}{a_1} - \frac{\bar{u}_1 a_1}{|a_1|^2 - 1} - \frac{\bar{u}_2 a_1}{a_1 \bar{a}_2 - 1}$$

$$t_{20} = -\frac{r u_1}{a_1^2} + \frac{|u_1|^2}{(|a_1|^2 - 1)^2} - \frac{u_1 \bar{u}_2}{(a_1 \bar{a}_2 - 1)^2},$$

$$t_{31} = \bar{q} + \frac{r}{a_2} - \frac{\bar{u}_2 a_2}{|a_2|^2 - 1} - \frac{\bar{u}_1 a_2}{a_2 \bar{a}_1 - 1},$$

$$t_{30} = -\frac{r u_2}{a_2^2} + \frac{|u_2|^2}{(|a_2|^2 - 1)^2} - \frac{u_2 \bar{u}_1}{(a_2 \bar{a}_1 - 1)^2}.$$

The area of D_{oil} is given by $A = t_{10} + t_{20} + t_{30}$.

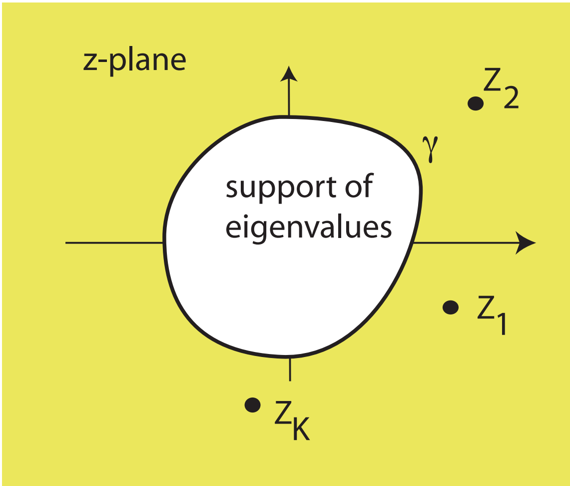
ASSOCIATED NORMAL MATRIX MODELS

$$z(p) = r\,p + q + \sum_{n=1}^{N_0} \frac{u_n}{p^n} + \sum_{i=1}^{K} \sum_{n=i}^{N_i} \frac{u_{i,n}}{(p-a_i)^n},$$

Dyson picture

Partition function of a 2D Coulomb plasma

$$\begin{cases} Z_N = \displaystyle\int e^{-E(z_1,\dots,z_N)} \prod_j \mathrm{d}z_j, \\ E := -\displaystyle\sum_{i>j} \log|z_i - z_j|^2 - \frac{1}{\hbar}\sum_j W(z_j) \quad \text{Energy}, \\ W(z) = -z\bar{z} + V(z) + \overline{V(z)} \quad \text{External field energy} \end{cases}$$



$$V_z(z) = \sum \text{Principal parts of } S(z) \text{ at } \infty, Z_1, \ldots, Z_K.$$

- Each point Z_i represents an external electrostatic source with a finite multipole expansion up to the order N_i.

- Positions, charges and multipole moments are functions of Whitham times.

Example

For a third-order pole Z_i

$$V(z) = Q_i \log(z - Z_i) + \frac{M_{i1}}{z - Z_i} + \frac{M_{i2}}{(z - Z_i)^2}$$

where

$$Z_i = -\tilde{t}_{i3}, \quad Q_i = -\tilde{t}_{i0},$$

$$M_{i1} = -\frac{2}{3}\tilde{t}_{i1}\,\tilde{t}_{i2},$$

$$M_{i2} = \frac{2}{9}\tilde{t}_{i2}^3.$$

Examples

N_i	$V(z)$	Interpretation
1	$Q_i^{(1)} \log(z - Z_i)$	point charge
2	$Q_i^{(2)} \log(z - Z_i) + \frac{M_{i1}^{(2)}}{z - Z_i}$	point charge+dipole
3	$Q_i^{(3)} \log(z - Z_i) + \frac{M_{i1}^{(3)}}{z - Z_i}$	point charge+dipole
	$+\frac{M_{i2}^{(3)}}{(z - Z_i)^2}$	+quadrupole

$$Z_i = -\tilde{t}_{i1}, \quad Q_i^{(1)} = -\tilde{t}_{i0},$$

$$Z_i = -\tilde{t}_{i2}, \quad Q_i^{(2)} = -\tilde{t}_{i0}, \quad M_{i1}^{(2)} = -\frac{1}{4}\tilde{t}_{i1}^2,$$

$$Z_i = -\tilde{t}_{i3}, \quad Q_i^{(3)} = -\tilde{t}_{i0}, \quad M_{i1}^{(3)} = -\frac{2}{3}\tilde{t}_{i1}\tilde{t}_{i2},$$

$$M_{i2}^{(3)} = \frac{2}{9}\tilde{t}_{i2}^3.$$